Chapter Outline

Trajectory Planning Models and Generation

Chapter 6
External Control and Trajectory Planning

6.1 External Control Fundamentals
6.2 Control Principles and Modes
6.3 Trajectory Models and Generation
6.4 Planning and Obstacle Avoidance
6.5 Tracking and Aggressive Flight Control
6.6 Frontier Control Technologies
6.7 Platform and Communication Interfaces
6.8 Examples and Experimental Validation
6.9 Exercises and Classroom Discussion

5 Units in This Section

6.3.1 Planning Objectives and Key Concepts
Textbook: Secs. 6.1.3.1, 6.1.3.2 (feasibility and optimality)

6.3.2 Problem Modeling and Objective Functions
Textbook: Sec. 6.1.3.3 (model and objective)

6.3.3 Constraints
Textbook: Sec. 6.1.3.3 (constraints)

6.3.4 Discretization and Basis Functions
Textbook: Sec. 6.1.3.3 (trajectory representation)

6.3.5 Trajectory Generation and Solution
Textbook: Secs. 6.1.3.3, 6.1.3.5 (process Step 3)

6.3.1 Planning Objectives and Key Concepts: Planning Objectives

Turn "reach the goal" into a motion task with clear checks

$$\mathbf{p}_0 = (0,0,-5)^{\mathsf{T}}\ \mathrm{m}, \quad \mathbf{p}_f = (10,0,-5)^{\mathsf{T}}\ \mathrm{m}$$

Before the mission: generate a reference → **During execution: replan online** → **After the mission: return or land**

Frame and initial state: north–east–down (NED), with down positive; 5 m above ground is −5 m. The vehicle initially hovers.

Task: move from $\mathbf{p}_0$ to $\mathbf{p}_f$ with no obstacles and stop at both ends; speed ≤3 m/s, acceleration ≤2 m/s².

The same arrival task requires different motion capabilities under different schedules.

6.3.1 Planning Objectives and Key Concepts: Key Concepts

What do path, trajectory, feasibility, and optimality describe?

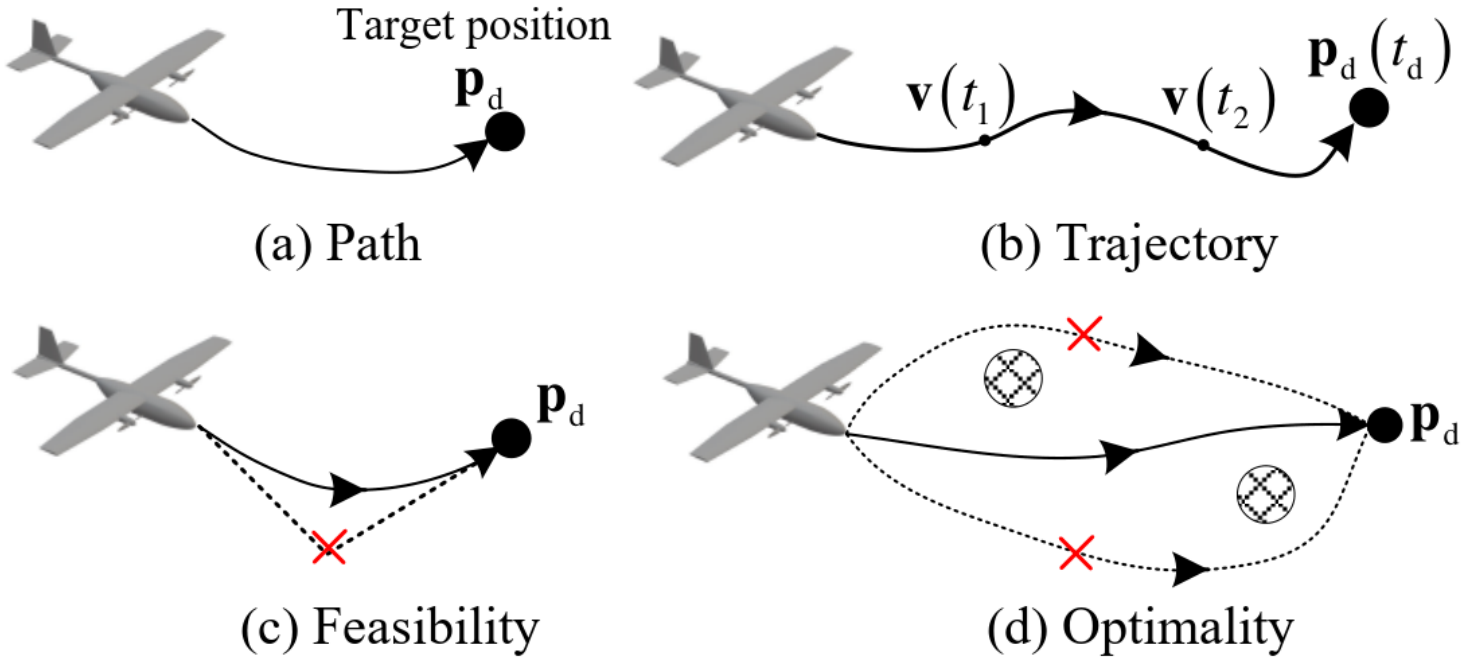


Fig. 6.5 Path: space; trajectory: space plus time. Feasible: constraints satisfied. Optimal: least cost among feasible trajectories.
Legend: red crosses—unachievable turns; crossed circles—obstacles; $\mathbf{p}_d$—goal position.

6.3.1 Planning Objectives and Key Concepts: Timing a Straight Line

Adding time to a straight path gives each reference point

$$\mathbf{r}(s) = \mathbf{p}_0 + s(\mathbf{p}_f - \mathbf{p}_0), \quad 0 \le s \le 1$$

$$s(t) = \frac{t}{T}, \quad T = 5\ \text{s}, \quad \mathbf{p}(t) = \mathbf{r}(s(t))$$

$$\mathbf{p}(2) = (4,0,-5)^{\mathsf{T}}\ \text{m}, \quad \mathbf{v} = (2,0,0)^{\mathsf{T}}\ \text{m/s}$$

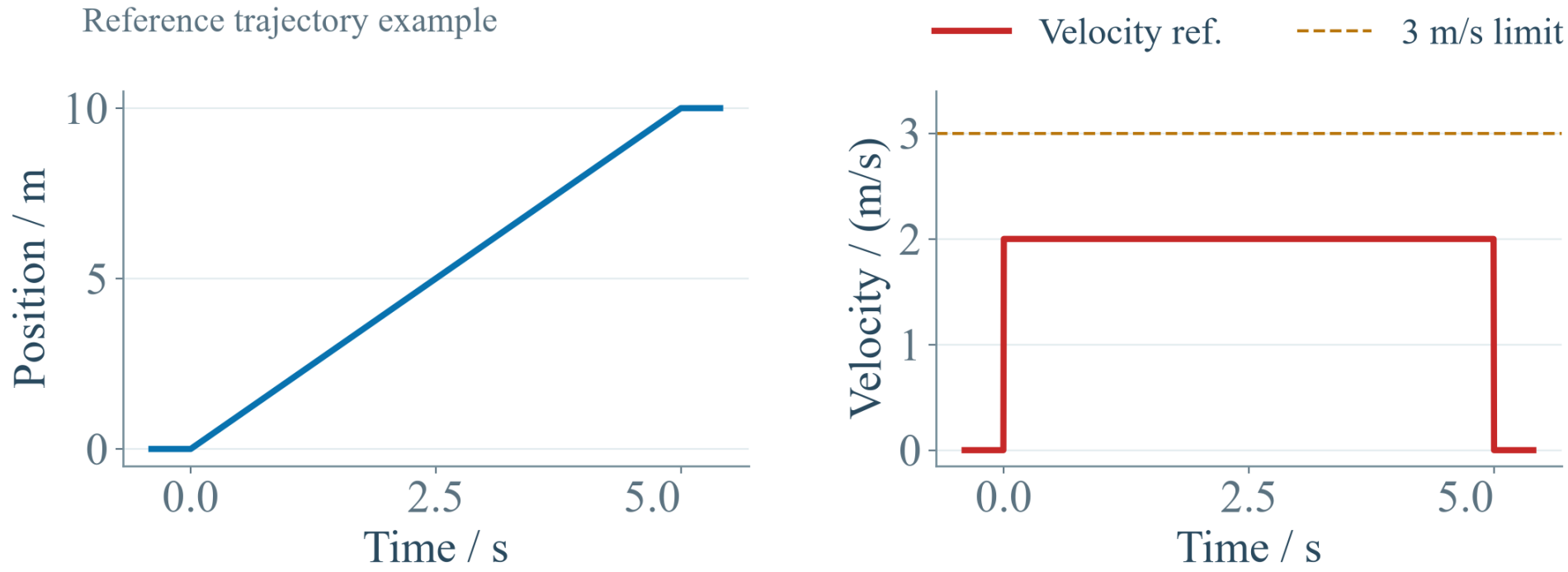
s: dimensionless progress; T: total duration (s); $\mathbf{r}$: path; $\mathbf{p}$: position reference (m).

Path following: move along a spatial curve without requiring a particular point at a particular time.

Trajectory tracking: follow the corresponding position and motion state at each specified time.

6.3.1 Planning Objectives and Key Concepts: Complete Reference

Cruise speed meets the limit, but start and stop are abrupt



Constant speed, 2 m/s, is below the 3 m/s limit. Endpoint velocity jumps between 0 and 2 m/s (vertical lines), which finite acceleration cannot realize.

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6.3.2 Problem Modeling and Objective Functions: Problem Model

Position and velocity form a 6D state; acceleration is the input

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{v}(t) \end{bmatrix} \in \mathbb{R}^6, \quad \mathbf{v} = \dot{\mathbf{p}}$$

$$\dot{\mathbf{p}} = \mathbf{v}, \quad \dot{\mathbf{v}} = \mathbf{u} \quad (\mathbf{u} = \mathbf{a})$$

$\mathbf{u} = \mathbf{a}$ ➡ Integrate ➡ $\mathbf{v}$ ➡ Integrate ➡ $\mathbf{p}$

$$\mathbf{x}_i(t) = \begin{bmatrix} \mathbf{p}_i(t) \\ \mathbf{v}_i(t) \end{bmatrix} \in \mathbb{R}^6, \quad i = 1, \ldots, N \qquad (6.1)$$

$\mathbf{p}, \mathbf{v}$: 3D position (m) and velocity (m/s); $\mathbf{u} = \mathbf{a}$: acceleration input (m/s²).
The 6D state $\mathbf{x}$ contains position and velocity; i is the vehicle index and N the number of vehicles.
Extend the model with attitude, angular velocity, and actuator constraints as the task requires.

6.3.2 Problem Modeling and Objective Functions: Objective Function

The objective compares costs; constraints determine acceptance

$$J[\mathbf{p}] = \int_0^T \left(k_1 \|\ddot{\mathbf{p}}(t)\|^2 + k_2 \|\mathbf{u}(t)\|^2\right) \mathrm{d}t$$

$$J^* = \min_{\{\mathbf{p}_i\}} \sum_{i=1}^{N} \int_0^T \left(k_1 \|\ddot{\mathbf{p}}_i\|^2 + k_2 \|\mathbf{u}_i\|^2 + \cdots\right) \mathrm{d}t \qquad (6.2)$$

Smoothness term: $\ddot{\mathbf{p}}$ is jerk, the rate of change of acceleration (m/s³).

Weights and units: $k_1, k_2 > 0$; reference scales can normalize the terms. J^* is the minimum cost, denoted J in textbook Eq. (6.2).

6.3.2 Problem Modeling and Objective Functions: Smoothness Metrics

Jerk and snap penalize changes of different orders

Quantity and Units	Position Derivative	Integral of the Square
Jerk	$\mathbf{p}^{(3)}(t)$	$J_3 = \int_0^T \|\mathbf{p}^{(3)}\|^2 \, \mathrm{d}t$
Units: m/s³	Rate of change of acceleration	Single segment: quintic; 6 endpoint conditions
Snap	$\mathbf{p}^{(4)}(t)$	$J_4 = \int_0^T \|\mathbf{p}^{(4)}\|^2 \, \mathrm{d}t$
Units: m/s⁴	Rate of change of jerk	Single segment: seventh degree; 8 endpoint conditions

Other objectives: time, coverage, path length. Check task fit, units, and conflicts.

6.3.2 Problem Modeling and Objective Functions: One Calculation

Compute a short-interval cost for a cubic candidate

$$x(t) = 0.08t^3, \quad 0 \le t \le 5\ \text{s}; \quad y = 0,\ z = -5$$

$$a_x(t) = 0.48t, \quad j_x(t) = 0.48$$

$$J = \int_0^5 \left[\left(\frac{j_x}{j_0} \right)^2 + \left(\frac{a_x}{a_0} \right)^2 \right] \mathrm{d}t$$

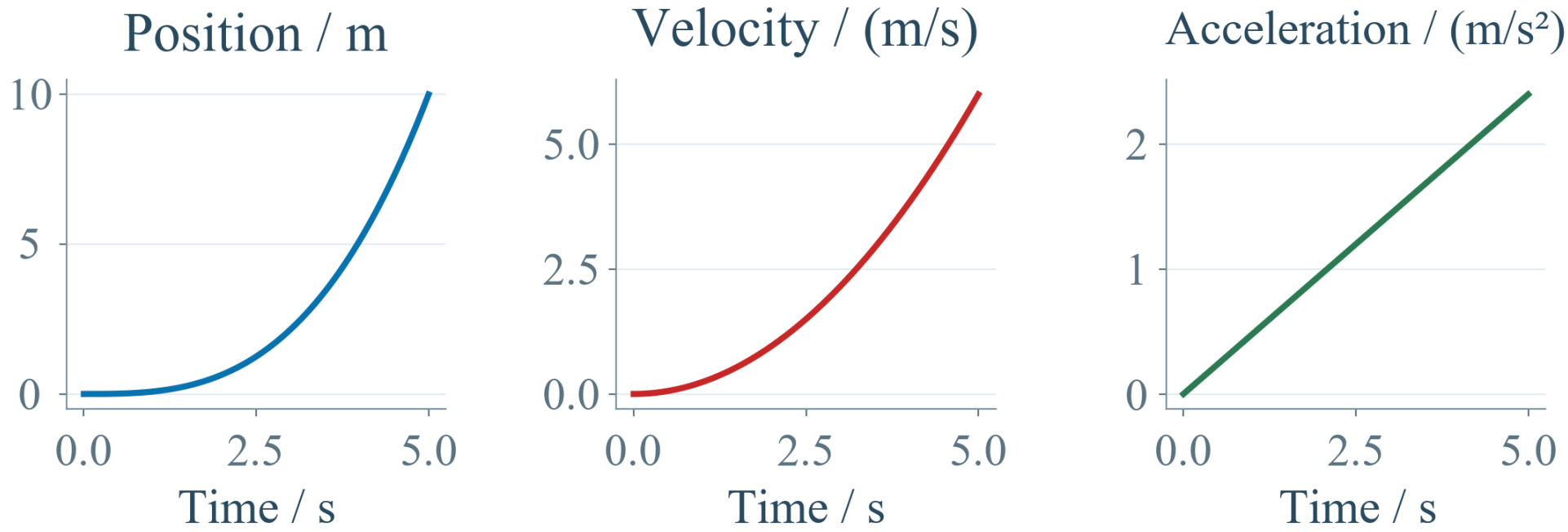
Units: north position x (m), time t (s), a_x (m/s²), j_x (m/s³).

Reference scales: $j_0 = 1\ \text{m/s}^3$, $a_0 = 1\ \text{m/s}^2$. Use a 0.1 s interval at 2 s.

$$\Delta J \approx (0.48^2 + 0.96^2) \times 0.1 = 0.1152\ \text{s}$$

6.3.2 Problem Modeling and Objective Functions: Complete Candidate

Integrating the cost does not replace the endpoint check



Complete candidate: $J = 1.152 + 9.6 = 10.752$ s; terminal velocity is 6 m/s, so the vehicle has not stopped.

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6.3.3 Constraints: Boundary Conditions

Specify how to arrive, as well as where to arrive

$$\mathbf{p}_i(0) = \mathbf{p}_i^{\text{start}}, \quad \mathbf{p}_i(T) = \mathbf{p}_i^{\text{goal}} \qquad (6.3)$$

North-Axis Condition	Start	Finish
Position	$x(0) = 0\ \text{m}$	$x(T) = 10\ \text{m}$
Velocity	$\dot{x}(0) = 0$	$\dot{x}(T) = 0$
Acceleration	$\ddot{x}(0) = 0$	$\ddot{x}(T) = 0$

Passing through and stopping at the goal require different boundary conditions.

6.3.3 Constraints: Motion Limits

Completion time changes velocity, acceleration, and jerk

$$\|\dot{\mathbf{p}}_i(t)\| \le v_{\max}, \quad \|\ddot{\mathbf{p}}_i(t)\| \le a_{\max} \qquad (6.5)$$

$$\tau = \frac{t}{T}, \quad \mathbf{p}(t) = \mathbf{r}(\tau), \quad 0 \le \tau \le 1$$

$$\mathbf{v} = \frac{\mathbf{r}'}{T}, \quad \mathbf{a} = \frac{\mathbf{r}''}{T^2}, \quad \mathbf{j} = \frac{\mathbf{r}'''}{T^3}$$

Dots denote derivatives with respect to time t; primes use normalized time τ; $\|\cdot\|$ denotes vector magnitude.

Here speed ≤3 m/s and acceleration ≤2 m/s² must hold over the entire interval.

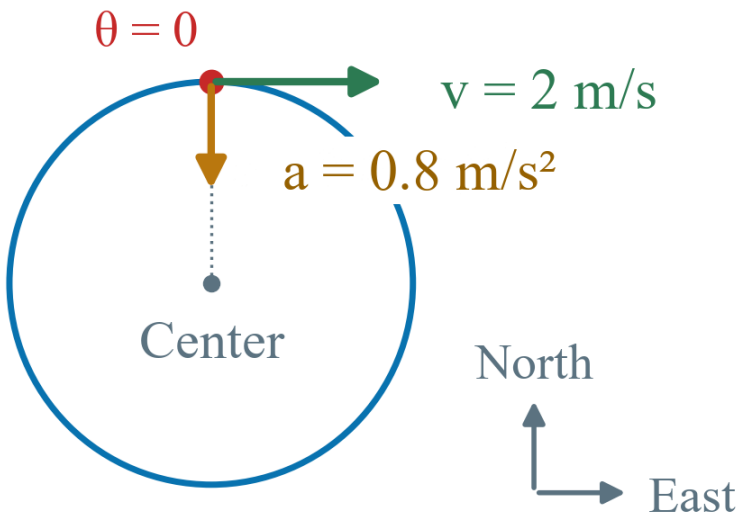
For the same curve, doubling T reduces peak velocity, acceleration, and jerk to 1/2, 1/4, and 1/8.

6.3.3 Constraints: Circular Trajectory Model

Circular velocity is tangential; acceleration points inward

$$\theta = \omega t, \quad \mathbf{p}(t) = \begin{bmatrix} R\cos\theta \\ R\sin\theta \\ -5 \end{bmatrix}, \quad R = 5\ \text{m}$$

$$\mathbf{v} = \begin{bmatrix} -R\omega\sin\theta \\ R\omega\cos\theta \\ 0 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} -R\omega^2\cos\theta \\ -R\omega^2\sin\theta \\ 0 \end{bmatrix}$$



Constant-speed arc: $\omega = 0.4$ rad/s; angle is measured north toward east.

$$\mathbf{v}(0) = (0,2,0)^{\mathsf{T}}\ \text{m/s}, \quad \mathbf{a}(0) = (-0.8,0,0)^{\mathsf{T}}\ \text{m/s}^2$$

Start-and-stop transitions also produce tangential acceleration; check their motion limits separately.

6.3.3 Constraints: Circular Motion Limits

At fixed radius, two limits jointly bound angular velocity

$$v = R|\omega|, \quad a = R\omega^2$$

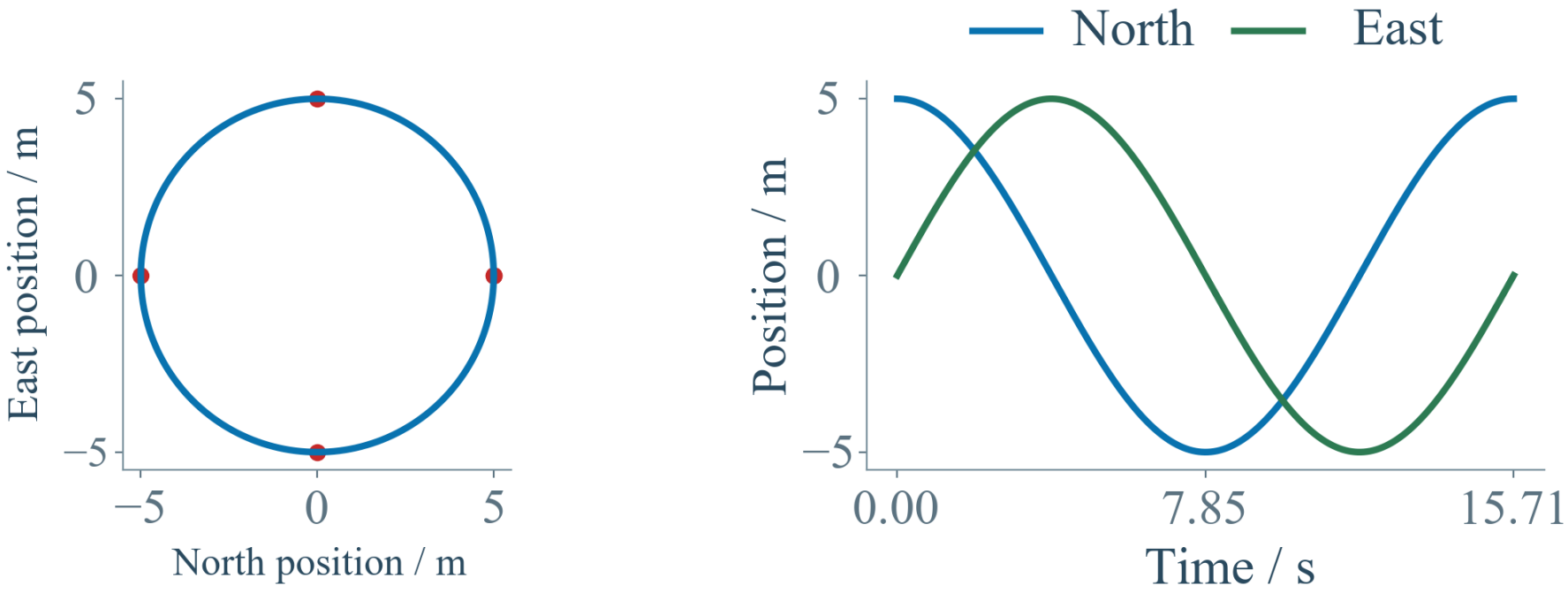
$$|\omega| \le \min\left(\frac{v_{\max}}{R}, \sqrt{\frac{a_{\max}}{R}}\right) = \min(0.6, \sqrt{0.4}) = 0.6 \text{ rad/s}$$

$$T_{\text{lap}} = \frac{2\pi}{|\omega|} \ge 10.472 \text{ s}$$

Angular Velocity	Speed	Acceleration	Interval Check
0.4 rad/s	2 m/s	0.8 m/s²	Meets both limits
0.8 rad/s	4 m/s	3.2 m/s²	Exceeds both limits

6.3.3 Constraints: Complete Circular Result

The spatial curve and time histories must both close



Red dots mark quarter laps; from (5,0), motion initially heads east. One lap takes 15.708 s; speed is 2 m/s and acceleration 0.8 m/s².

6.3.3 Constraints: Turning and Nonholonomic Constraints

Motion constraints also depend on vehicle type

$$a_{\mathrm{n}} = v^2\kappa = \frac{v^2}{R}, \quad R \geq R_{\min}$$

a_n: normal acceleration (m/s²)
$\kappa = 1/R$: curvature (1/m)
$R_{\min}$: minimum turning radius (m)

Multicopter	Ideal Wheeled Vehicle	Fixed-Wing Aircraft
Motion Conditions	Motion Conditions	Motion Conditions
Thrust, tilt, climb capability	$-\dot{x}\sin\psi + \dot{y}\cos\psi = 0$	Minimum airspeed, turn radius, climb angle
Trajectory Implications	Trajectory Implications	Trajectory Implications
Can hover; acceleration is bounded	No slip: motion along heading. ψ: heading, north toward east	Stopping and reversing cannot replace a turn

Nonholonomic constraints restrict velocity directions and cannot be integrated into position constraints; a wheeled vehicle's no-slip condition is one example.

6.3.3 Constraints: Collision Avoidance

Multi-vehicle separation constraints are usually nonconvex

$$\|\mathbf{p}_i(t) - \mathbf{p}_j(t)\| \geq d_{\min}, \quad i \neq j, \quad t \in [0, T] \qquad (6.4)$$

1D Relative Position	Check Against 2 m Separation	Feasible?
+2 m	Distance: 2 m	Yes
−2 m	Distance: 2 m	Yes
Their average: 0 m	Distance: 0 m	No

Convex set: every point on a segment joining any two members stays in the set. Their average is a convex combination.

Here the average leaves the feasible set, so the constraint is nonconvex. Recheck the original constraint after solving.

6.3.3 Constraints: Continuous-Interval Checks

Checking sample times alone can miss a collision

$$p_1(t) = -1 + 2t, \quad p_2(t) = 1 - 2t, \quad 0 \le t \le 1$$



Vehicle positions: −1, +1 m Separation: 2 m **Safe endpoint**	Vehicle positions: 0, 0 m Separation: 0 m **Collision within the segment**	Vehicle positions: +1, −1 m Separation: 2 m **Safe endpoint**

Passing checks at discrete nodes still requires verifying motion and separation between nodes.

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6.3.4 Discretization and Basis Functions: Discretization

Optimize nodes; use differences to link motion states

$$\{t_k, \mathbf{p}_1(t_k), \ldots, \mathbf{p}_N(t_k)\}_{k=0}^{K} \qquad (6.6)$$

K: number of intervals, giving $K+1$ nodes. For one vehicle, $\mathbf{p}_k = \mathbf{p}(t_k)$ with uniform spacing $\Delta t = t_{k+1} - t_k$.

$$\mathbf{v}_k \approx \frac{\mathbf{p}_{k+1} - \mathbf{p}_k}{\Delta t}, \quad \mathbf{a}_k \approx \frac{\mathbf{p}_{k+1} - 2\mathbf{p}_k + \mathbf{p}_{k-1}}{\Delta t^2}$$

Example: positions 0, 0.1, 0.4 m at 0.1 s intervals give velocities 1 and 3 m/s.

$$a_1 \approx \frac{0.4 - 2 \times 0.1 + 0}{0.1^2} = 20 \ \mathrm{m/s^2}$$

Positions differ by just 0.1 and 0.3 m, yet acceleration reaches 20 m/s², far above the 2 m/s² limit.

6.3.4 Discretization and Basis Functions: Basis Representation

Optimize trajectory coefficients directly

$$\mathbf{p}_i(t) = \sum_{j=0}^{M} \phi_j(t)\mathbf{A}_{i,j} = \mathbf{A}_i\boldsymbol{\Phi}(t) \qquad (6.7)$$

$$\mathbf{A}_{i,j} = \begin{bmatrix} \alpha_{i,j} \\ \beta_{i,j} \\ \gamma_{i,j} \end{bmatrix}, \quad \mathbf{A}_i \in \mathbb{R}^{3\times(M+1)}$$

$$\boldsymbol{\Phi}(t) = [\phi_0(t), \dots, \phi_M(t)]^{\mathsf{T}}$$

$$\mathbf{v}_i = \mathbf{A}_i\dot{\boldsymbol{\Phi}}, \quad \mathbf{a}_i = \mathbf{A}_i\ddot{\boldsymbol{\Phi}}$$

ϕ_j: selected basis function j, with $M+1$ functions in total; $\boldsymbol{\Phi}$: basis-function column vector.
$\mathbf{A}_i$: unknown coefficient matrix for vehicle i; each column $\mathbf{A}_{i,j}$ has north, east, and down coefficients.
Once the coefficients are known, position, velocity, and acceleration can be evaluated at any time.

6.3.4 Discretization and Basis Functions: Basis Families

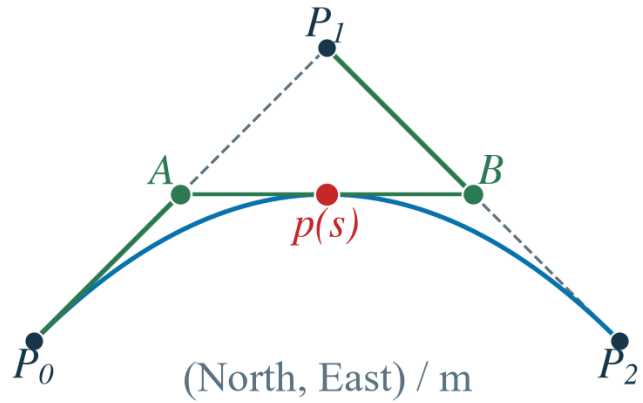
The chosen basis changes representation and constraints

Basis Family	Main Parameters	Properties and Checks
Polynomials	Coefficients of each power	Analytic derivatives; check scaling and junction continuity
Bézier curves	Control points and degree	Intuitive shape control; control points usually are not waypoints
B-splines	Control points, degree, knot vector	Local support: a control point affects nearby segments. Repeated knots reduce continuity.
Fourier basis	Sine and cosine coefficients	Periodic representation; check periodic boundaries
Radial basis functions (RBFs)	Centers, widths, weights	Shape depends on distance from centers

6.3.4 Discretization and Basis Functions: Control-Point Example

Bézier control points shape the curve; it need not pass through them

$$\mathbf{p}(s) = (1-s)^2\mathbf{P}_0 + 2s(1-s)\mathbf{P}_1 + s^2\mathbf{P}_2$$



Parameter $s = 0.50$

First level

$A = (2.5, 2.5)$

$B = (7.5, 2.5)$

Second level

$p = (5.0, 2.5)$

Three control points:

$\mathbf{P}_0 = (0,0)$

$\mathbf{P}_1 = (5,5)$

$\mathbf{P}_2 = (10,0)$

Dashed lines: control polygon. The curve lies in the control points' convex hull; keeping the entire hull clear of obstacles guarantees geometric clearance.

$$\mathbf{p}(0.5) = \frac{1}{4}(0,0) + \frac{1}{2}(5,5) + \frac{1}{4}(10,0) = (5,2.5)$$

6.3.4 Discretization and Basis Functions: Polynomial Basis

Degree seven has eight coefficients; normalize long durations

$$x_i(t) = \alpha_{i,0} + \alpha_{i,1}t + \cdots + \alpha_{i,6}t^6 + \alpha_{i,7}t^7 \qquad (6.8)$$

$$\tau = \frac{t - t_0}{T}, \quad x_i(t) = \sum_{j=0}^{7} c_{i,j}\tau^j, \quad 0 \le \tau \le 1$$

$$30^7 = 2.187 \times 10^{10}, \quad 0 \le \tau^7 \le 1$$

t_0: segment start time; T: segment duration (s); $c_{i,j}$: normalized coefficients (m).
Original coefficients $\alpha_{i,j}$ have units $\mathrm{m/s}^j$; the scale example uses a 30 s segment.
Why degree seven? A single minimum-snap segment has 8 coefficients set by endpoint position, velocity, acceleration, and jerk.

6.3.4 Discretization and Basis Functions: Variable Counts

For this task, nodes greatly outnumber basis coefficients

Representation and Counting	Raw Scalar Count	After Fixing Both End Positions
30 intervals, 31 nodes	$10 \times 31 \times 3 = 930$	$930 - 60 = 870$
Exactly 30 nodes	$10 \times 30 \times 3 = 900$	$900 - 60 = 840$
One degree-seven segment per axis	$10 \times 8 \times 3 = 240$	$240 - 60 = 180$

Example: 10 vehicles, 3D position, fixed timing; count only position nodes or coefficients. Endpoint constraints may be retained or eliminated. Do not mix raw counts with counts after elimination.

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6.3.5 Trajectory Generation and Solution: Quintic Derivation

Six endpoint conditions uniquely determine a quintic segment

$$x(t) = 10q(\tau), \quad q(\tau) = \sum_{j=0}^{5} c_j \tau^j, \quad \tau = t/T$$

$$q(0) = q'(0) = q''(0) = 0$$

$$q(1) = 1, \quad q'(1) = q''(1) = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 12 & 20 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_0 = c_1 = c_2 = 0, \quad (c_3, c_4, c_5) = (10, -15, 6)$$

$$q(\tau) = 10\tau^3 - 15\tau^4 + 6\tau^5$$

q: normalized displacement; c_j: its coefficients, all dimensionless. x is in m and T in s.

Optimality: with fixed T and endpoint position, velocity, and acceleration, this quintic minimizes the integral of squared jerk.

6.3.5 Trajectory Generation and Solution: Quintic Calculation

Derive position, velocity, and acceleration from one polynomial

$$q = 10\tau^3 - 15\tau^4 + 6\tau^5, \quad x = 10q$$

$$q' = 30\tau^2(1-\tau)^2, \quad v_x = \frac{10}{T}q'$$

$$q'' = 60\tau - 180\tau^2 + 120\tau^3, \quad a_x = \frac{10}{T^2}q''$$

Substitute $T = 5$ s, $t = 1.25$ s, $\tau = 0.25$:

$$x = 1.03515625\ \text{m}, \quad v_x = 2.109375\ \text{m/s}, \quad a_x = 2.25\ \text{m/s}^2$$

6.3.5 Trajectory Generation and Solution: Duration Selection

Analytic peaks set a minimum duration for this quintic

$$v_{\text{peak}} = \frac{1.875D}{T}, \quad a_{\text{peak}} = \frac{10D}{\sqrt{3}\,T^2}, \quad D = 10\ \text{m}$$

Derivation: $q'(0.5) = 1.875$; when $q''' = 0$, $\tau = (3 \pm \sqrt{3})/6$ and $|q''|_{\max} = 10/\sqrt{3} \approx 5.774$.

$$T \geq \max\left(\frac{1.875D}{v_{\max}}, \sqrt{\frac{10D}{\sqrt{3}\,a_{\max}}}\right)$$

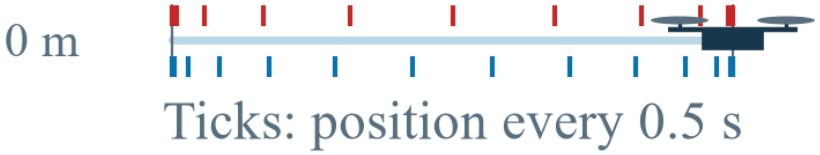
$$T \geq \max(6.25, 5.373)\ \text{s} = 6.25\ \text{s}$$

Choose 6.5 s: speed 2.885 m/s and acceleration 1.367 m/s² meet both limits.
Cost: $J \approx 12.45$ s, above the infeasible cubic's 10.75 s. Filter by constraints before comparing costs.

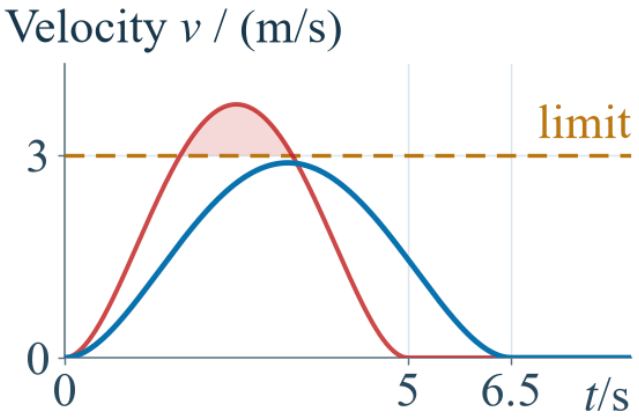
6.3.5 Trajectory Generation and Solution: Time Scaling

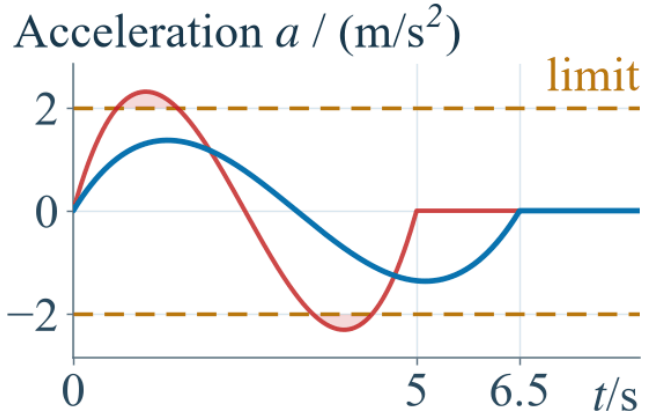
Longer duration preserves the path and lowers motion peaks

Lower bound $T \geq$ max(6.25, 5.373) s; take 6.5 s



T	5	→ 6.5 s
v_{peak}	3.75	→ 2.885
a_{peak}	2.309	→ 1.367





Animation: same path, 5→6.5 s; analytic values.

Same path: both plans move 0→10 m and stop. Only position versus time changes.

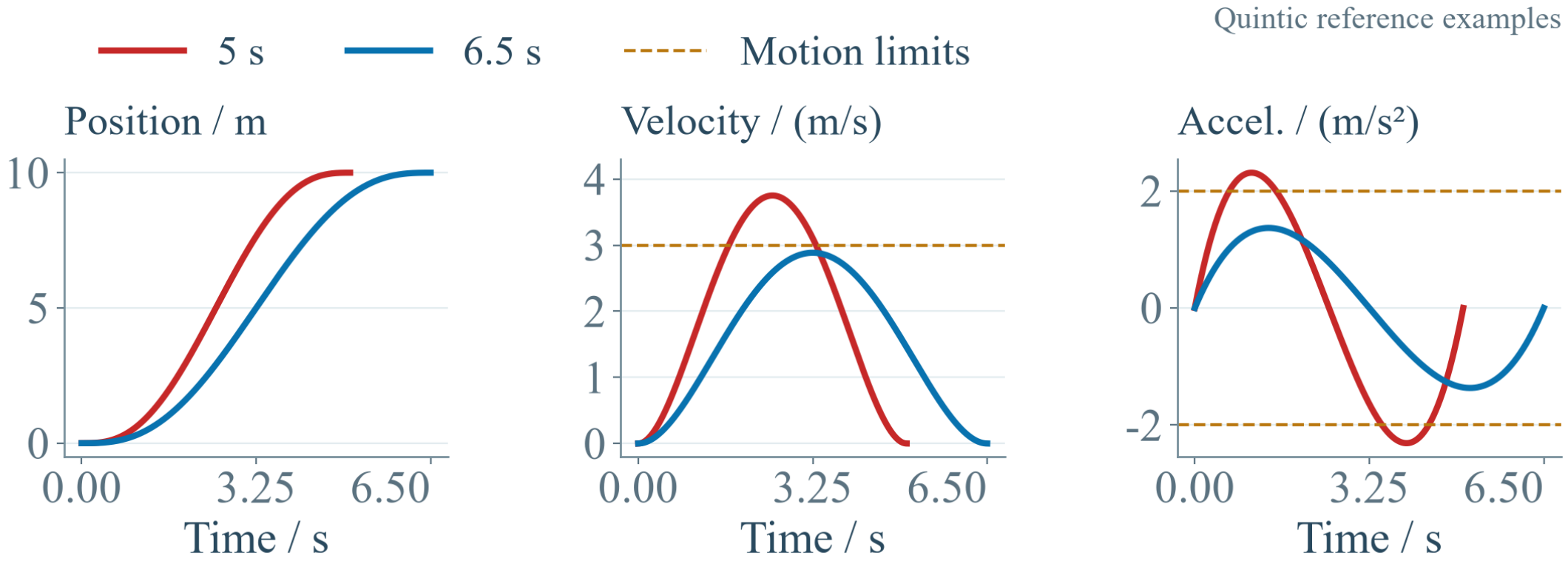
Peak scaling: speed is inversely proportional to T; acceleration to T^2.

Minimum duration: 5.373 s for acceleration, 6.25 s for speed. Choose 6.5 s for margin.

6.3.5 Trajectory Generation and Solution: Complete Quintic Result

Extending the same path to 6.5 s meets both motion limits

Quintic reference examples



Red: 5 s; blue: 6.5 s. Orange dashed limits: 3 m/s and 2 m/s². Blue peaks: 2.885 m/s and 1.367 m/s².

6.3.5 Trajectory Generation and Solution: Piecewise Generation

Multiple waypoints require segment and junction conditions

Waypoint	Position (North, East) / m	Arrival Time	Passage Condition
Start	(0,0)	0 s	At rest
Middle	(5,3)	4.2 s	Stop, then depart
Finish	(10,0)	8.4 s	At rest

$$\mathbf{p}(t) = \mathbf{P}_\ell + (\mathbf{P}_{\ell+1} - \mathbf{P}_\ell)\, q\left(\frac{t - t_\ell}{4.2}\right)$$

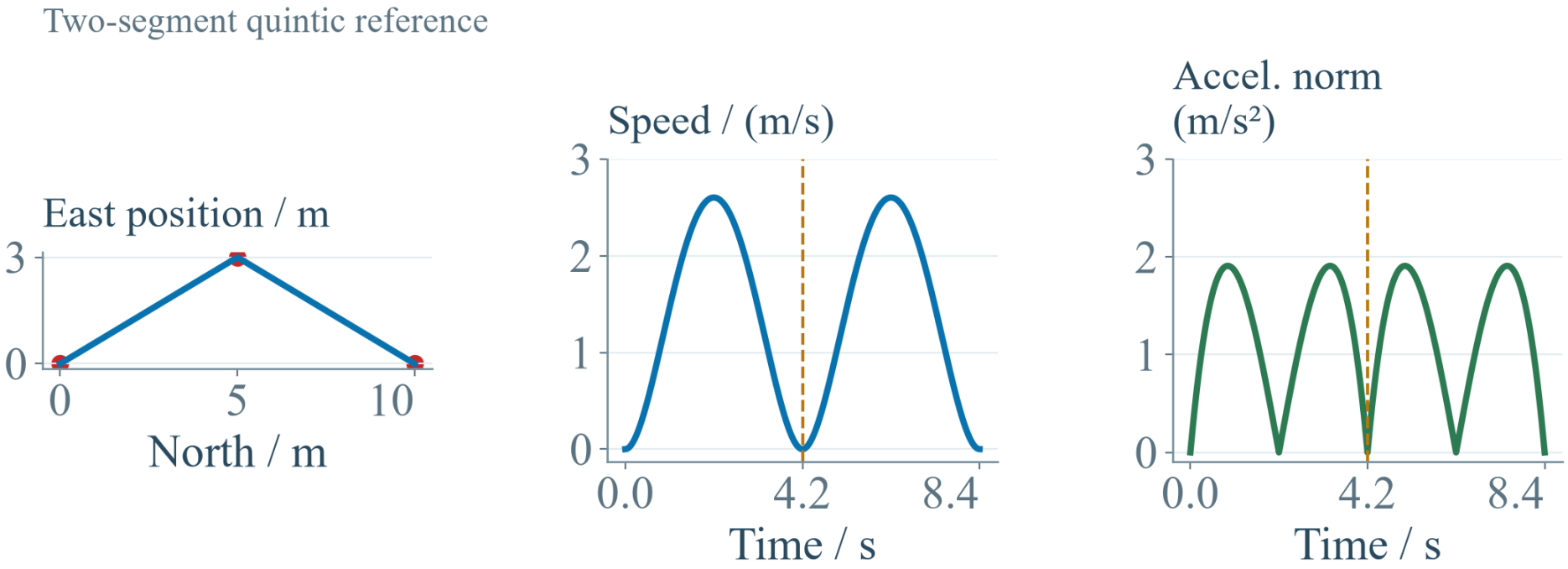
$\ell = 0,1$: segment index; $t_0 = 0$ s, $t_1 = 4.2$ s: segment start times.

$\mathbf{P}_\ell$: 3D waypoints, all with vertical coordinate −5 m; position, velocity, and acceleration are continuous at the junction.

Segment length $\sqrt{34} \approx 5.83$ m gives $T \geq 4.10$ s from the duration bound; choose 4.2 s.

6.3.5 Trajectory Generation and Solution: Complete Waypoint Result

Check the waypoint stop and each segment's peaks



Waypoint stop: 4.2 s. Peak speed: 2.603 m/s; peak acceleration: 1.908 m/s².

6.3.5 Trajectory Generation and Solution: Solving and Checking

Fixed timing and bases often give a convex quadratic program

$$\min_{\mathbf{c}} \ \frac{1}{2}\mathbf{c}^{\mathsf{T}}\mathbf{H}\mathbf{c} + \mathbf{f}^{\mathsf{T}}\mathbf{c}, \quad \mathbf{H} \succeq 0 \qquad \mathbf{l} \leq \mathbf{A}\mathbf{c} \leq \mathbf{u}$$

$\mathbf{c}$: unknown coefficients; $\mathbf{H}, \mathbf{f}$: known quadratic matrix and linear coefficients.

$\mathbf{A}$: linear constraint matrix; $\mathbf{l}, \mathbf{u}$: lower/upper bounds; $\mathbf{H} \succeq 0$: positive semidefinite (convex cost).

Here $\mathbf{u}$ is a constraint upper bound, distinct from the double-integrator input.

Timing and Basis → **Formulate and Solve** → **Continuous Constraints** → **Output Motion Reference**

Quadratic programming (QP): a quadratic cost in the coefficients with linear constraints; solve with OSQP (Operator Splitting Quadratic Program). A speed-norm upper bound is convex but nonlinear; a pairwise-distance lower bound is nonconvex. Treat them separately.

A global route can first be found by discrete search, then timed and smoothed with local basis functions. Output position, velocity, and acceleration (PVA).

6.3.5 Trajectory Generation and Solution: Reading an Example

The installed line example sends interpolated points to the API

Location: [installation directory]/RflySimAPIs/6.RflySimExtCtrl/2.AdvExps/e4.TrajPlanExps/1.LineTrajMotion/LineWPCtrl.py

```
start = (0, 0, -10); end = (20, 15, -10); num_points = 100
for i in range(num_points):
    t = i / (num_points - 1)
    x = start[0] + (end[0] - start[0]) * t
    mav.SendPosNED(x, y, z, yaw)
    time.sleep(0.033)
```

t interpolates from 0 to 1. At 0.033 s per point, 100 points cover 25 m in about 3.3 s (nominally 7.6 m/s). Endpoint velocity jumps as in the linear reference. Execution: Section 6.8.2, “Running and Acceptance.”

$$i = 33: \quad s = \frac{33}{99} = \frac{1}{3}, \quad \mathbf{p} = (6.667, 5, -10)^{\mathsf{T}}\ \mathrm{m}$$

6.3.5 Trajectory Generation and Solution: Derivative Consistency

Circle example: default direction=−1 gives inconsistent north velocity

Source: [installation directory]/RflySimAPIs/6.RflySimExtCtrl/
2.AdvExps/e4.TrajPlanExps/3.CirularTrajMotion/UAVCtrl.py

Consistent formulation: d: direction; ω: angular velocity (rad/s); t: elapsed time (s).

$$\theta = d\omega t, \quad d \in \{-1, +1\}, \quad x = R\cos\theta\,, \quad y = R\sin\theta$$

$$v_x = -Rd\omega\sin\theta\,, \quad v_y = Rd\omega\cos\theta$$

Source defaults: $d = -1$, $R = 6\,\mathrm{m}$, $\omega = 0.4\,\mathrm{rad/s}$.
At angle = $\pi/2$, the derivative of the printed north position is -2.4 m/s, but the sent velocity is +2.4 m/s.

Source: angle = ω*elapsed; vx = -R*ω*sin(angle)*direction; printed x = R*cos(angle)
See Section 6.5: “Circular Trajectory,” “Reading the Example,” “Code Counterexample.” Run criteria: Section 6.8.2, “Expectations and Troubleshooting.”

Classroom Exercises

Use new data to check counts, motion limits, and interval safety

Question	Key Points
For 20 vehicles in 3D with 50 time intervals, how many position scalars?	51 nodes: $20 \times 51 \times 3 = 3060$. Fixing both endpoint positions leaves $3060 - 120 = 2940$.
Circle: radius 2 m, speed limit 3 m/s, acceleration limit 2 m/s². Maximum angular velocity?	Speed gives 1.5 rad/s; acceleration gives 1.0 rad/s. Choose 1.0 rad/s; acceleration is the active constraint here.
15 m quintic start-and-stop motion in 6 s: does it meet the limits? Minimum duration?	Peak speed 4.69 m/s and acceleration 2.41 m/s² both exceed the limits. The duration formula requires at least 9.375 s.
Two vehicles cross the origin in 1 s at constant speed along north/east axes, starting/ending 2 m away. Do endpoints alone ensure ≥2 m separation?	No. Endpoint separation is 2.83 m, but both occupy the origin at 0.5 s, giving zero separation. Check the continuous interval.

Section Summary

Trajectory generation produces time functions that have been checked

Model: position, time, boundary states, motion capability, and environmental requirements jointly define the problem.

Solve: decide whether nodes or coefficients are unknowns; check variable counts and constraint types.

Verify: check endpoint conditions, derivative peaks, continuity, and separation across the full horizon, then output PVA.

Common errors: feasible does not mean optimal; smooth does not mean within limits; a reference is not actual motion. K intervals have K+1 nodes; safe nodes still require interval checks.

Trajectory modeling yields a computable reference; planning and closed-loop execution each still have their own tasks.