Chapter Outline

Sensor Calibration, Filtering, and State Estimation

Chapter 5
Filtering, State Estimation,
and Low-Level Control

5.1 Filtering and Low-Level Control

5.2 Sensor Calibration and Estimation

5.3 Controller Design and Allocation

5.4 Control Performance and Tuning

5.5 PX4 Architecture and Interfaces

5.6 Control Development and Testing

5.7 Other Vehicles and Advanced Cases

5.8 Exercises and Classroom Discussion

5 Units in This Section

5.2.1 Sensor Calibration and Measurement Model
Textbook: Sec. 5.1.3.2, Part 1; Sec. 5.3.1

5.2.2 Complementary Filter
Textbook: Sec. 5.1.3.2, Part 2; Sec. 5.3.1

5.2.3 Kalman Filter
Textbook: Sec. 5.1.3.2, Part 3

5.2.4 Extended Kalman Filter
Textbook: Sec. 5.1.3.2, Part 4; Sec. 5.3.1

5.2.5 Filter Comparison and Validation
Textbook: Sec. 5.1.3.2, Part 5

5.2.1 Sensor Calibration and Measurement Model: Attitude Measurement

States, Measurements, and Measurement Errors

An inertial measurement unit (IMU) provides angular velocity and specific force; the controller needs states such as attitude.

True State → **Sensor Measurements** → **Correction and Fusion** → **State Estimate**

Term	Question It Answers	Example in Attitude Estimation
State	How the vehicle is moving	Pitch angle, gyroscope bias
Measurement	What the sensor actually outputs	Three-axis angular velocity and specific force
Error	Why measurements differ from the model	Bias, scale factor, random noise, misalignment

5.2.1 Sensor Calibration and Measurement Model: Estimator Inputs and Outputs

From Six-Axis Measurements to Control Feedback

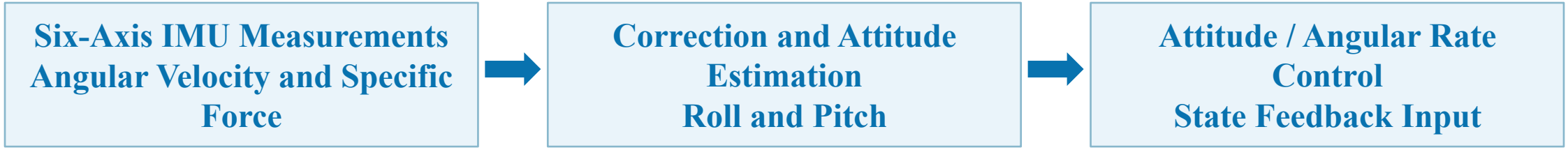


Figure: Estimation branch in textbook Fig. 5.2 (sensor measurements and state feedback are distinct signals)

Quantity	Input / Output Meaning	Effect on the Controller
Angular velocity measurement	rad/s; remove bias and retain body-axis conventions	Angular rate feedback is not an attitude angle
Acceleration measurement	m/s^2; specific force, not motion acceleration	Gravity-direction reference under quasi-static conditions
Estimate and data age	Attitude in rad; covariance and sampling time	Check confidence and delay before closing the loop

5.2.1 Sensor Calibration and Measurement Model: Error Model

Bias and Scale Factor in a One-Dimensional Measurement Model

The reading combines scaling, offset, and noise

$$y = sx + b + n$$

With known calibration parameters, subtract bias before dividing by the scale factor

$$\hat{x} = \frac{y - b}{s} = \frac{10.2 - 0.1}{1.01} = 10.0 \ \mathrm{m/s^2}$$

- x: true value; y: reading; b: bias; n: noise, all in m/s².
- s: dimensionless scale factor determining the response slope.
- Bias shifts the entire response; random noise remains after deterministic correction.

5.2.1 Sensor Calibration and Measurement Model: Three-Axis Model

Cross-Axis Coupling and Mounting Orientation Affect Measurements

$$\mathbf{y} = \mathbf{S}\mathbf{x} + \mathbf{b} + \mathbf{n}$$

Error	Physical Meaning	Required Identification Information
Three-axis bias	A fixed offset on each axis	Known attitude or reference to identify bias
Scale factors and cross-axis coupling	Unequal slopes and contributions from other axes	Coverage of multiple attitudes and directions
Mounting misalignment	Sensor axes differ from body axes	Direction reference or external attitude reference

S is a 3×3 matrix and **b** a 3×1 vector. One stationary data segment cannot readily identify all parameters.

5.2.1 Sensor Calibration and Measurement Model: Static and Dynamic Calibration

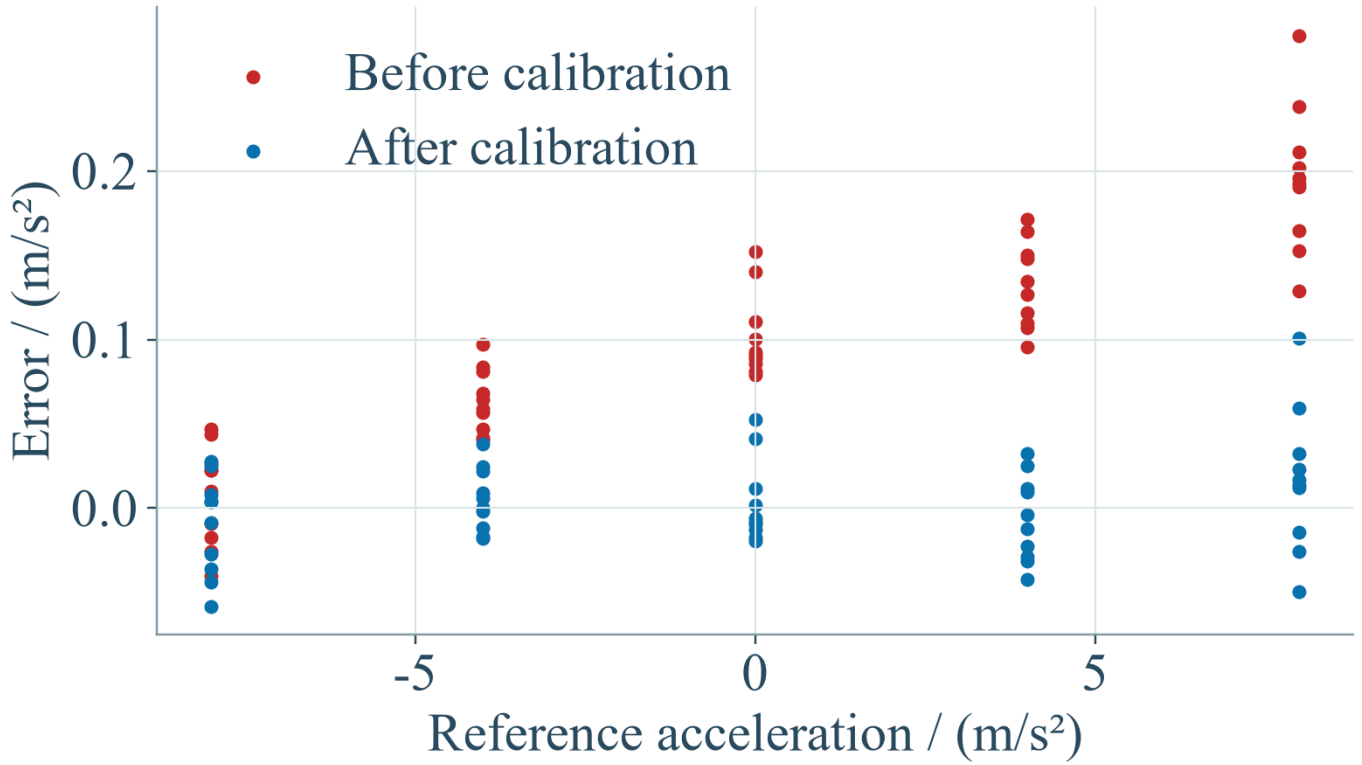
Data Coverage Determines Which Errors Can Be Identified

Task	Data and Reference	What to Validate
Accelerometer Static Calibration	Stationary data at multiple gravity orientations	Bias, scale factors, and cross-axis errors
Dynamic calibration	Known trajectory or external motion reference	Lag, frequency response, and motion-related errors
Magnetometer calibration	Magnetic measurements in multiple directions	Hard-iron bias, soft-iron scaling and coupling

- Stationary gyroscope data mainly estimate zero-rate bias; scale and cross-axis errors need a known angular motion reference.
- Fit on training data; assess errors on independent held-out data.
- Poor attitude coverage, timestamp misalignment, or incorrect units make small residuals hard to interpret.

5.2.1 Sensor Calibration and Measurement Model: Held-Out Validation

Independent Data Validate the Correction



Fit: 160 points; held out: 50.
Scale factor: 1.01000
Bias: 0.0987 m/s²
Root mean square error (RMSE):
Before: 0.122 m/s²
After: 0.029 m/s²
Conclusion: Errors before correction are mostly positive and grow with the reference value, indicating both bias and scaling error. Held-out errors cluster near 0 after correction.

Figure: Errors before and after correction at held-out attitudes (teaching data)

5.2.1 Sensor Calibration and Measurement Model: Model Validation

The Compensation Sign Depends on the Implementation

Example folder: [安装目录]/RflySimAPIs/5.RflySimFlyCtrl/1.BasicExps/e3-SensorCalib/
Entry: Readme.pdf (pages 2, 7–9); offline calculation: e3.1/calLM.m

$$\mathbf{a}_{\text{cal}} = \mathbf{K}_a(\mathbf{a}_{\text{raw}} + \mathbf{b}_a)$$

$$k = 1/s, \quad b_a = -b, \quad k(y + b_a) = (y - b)/s$$

- **Run:** In MATLAB, open e3.1, confirm e_acc_A.bin is there, and run calLM.m.
- **Expected:** Sample, Levenberg-Marquardt (LM) iteration, and before/after plots; scale factors and compensation printed.
- **Criterion:** calFunc.m-corrected magnitudes approach the gravity reference; check axes on held-out data.
- **Troubleshoot:** Match filename and CAL_ACC_SCALE/OFF to collection settings; check units and attitude coverage.

Correct units, coordinates, and compensation signs are prerequisites for filter inputs.

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Textbook: Sec. 5.1.3.2, Part 5

5.2.2 Complementary Filter: Sources of Attitude Information

Why Gyroscope and Gravity Information Complement Each Other

θ_{prev}: previous attitude; ω: angular velocity; b_g: zero-rate bias; Δt: sampling interval.

$$\theta_{\text{gyro}} = \theta_{\text{prev}} + (\omega - b_g)\Delta t \qquad \text{Eq. 5.2}$$

Channel	Direct Measurement	Attitude Information
Gyroscope	Angular velocity, rad/s	Integrates short-term angle changes; bias causes drift
Accelerometer	Specific force, m/s²	Gravity reference for roll and pitch under quasi-static conditions

Δt is the sampling interval in s. General three-dimensional attitude requires attitude kinematics; body angular rates differ from Euler angle derivatives.

5.2.2 Complementary Filter: Specific Force and Coordinates

Why an Accelerometer Has Nonzero Output at Rest

$$ {}^{b}\mathbf{a}_{\text{acc}} \approx {}^{b}\mathbf{a}_{\text{motion}} - \mathbf{R}_{e}^{b} \, {}^{e}\mathbf{g} \qquad \text{Eq. 5.3} $$

Condition	Motion Acceleration	Ideal Specific Force
Level and stationary	$\mathbf{0}$	$[0,0,-9.8]^{\top}\ \text{m/s}^2$
Free fall	$\mathbf{g}$	Approximately zero
Horizontal acceleration	$a_x = 2\ \text{m/s}^2$	$[2,0,-9.8]^{\top}\ \text{m/s}^2$

The earth frame is north-east-down (NED); body axes point forward, right, down. $\mathbf{R}_{e}^{b}$ maps earth-frame vectors into the body frame.

5.2.2 Complementary Filter: Accelerometer Attitude

Gravity Projections Provide a Pitch Reference

$$\theta_{\text{acc}} = \arctan\left(\frac{a_x}{\sqrt{a_y^2 + a_z^2}}\right) = atan2\left(a_x, \sqrt{a_y^2 + a_z^2}\right) \qquad \text{Eq. 5.4}$$

$$\mathbf{a} = [4.9, 0, -8.487]^{\top}\ \text{m/s}^2, \quad \theta_{\text{acc}} = atan2(4.9, 8.487) = 30^\circ$$

- Assume corrected measurements, quasi-static motion, and the conventional $\pm 90^\circ$ pitch range.
- atan2 improves handling of division by zero but cannot identify inverted attitude from this expression alone.
- Horizontal acceleration of 2 m/s² also produces about 11.5°; fuse other information during maneuvers.

5.2.2 Complementary Filter: False Pitch During Maneuvers

Specific Force Alone Cannot Distinguish Tilt from Horizontal Acceleration

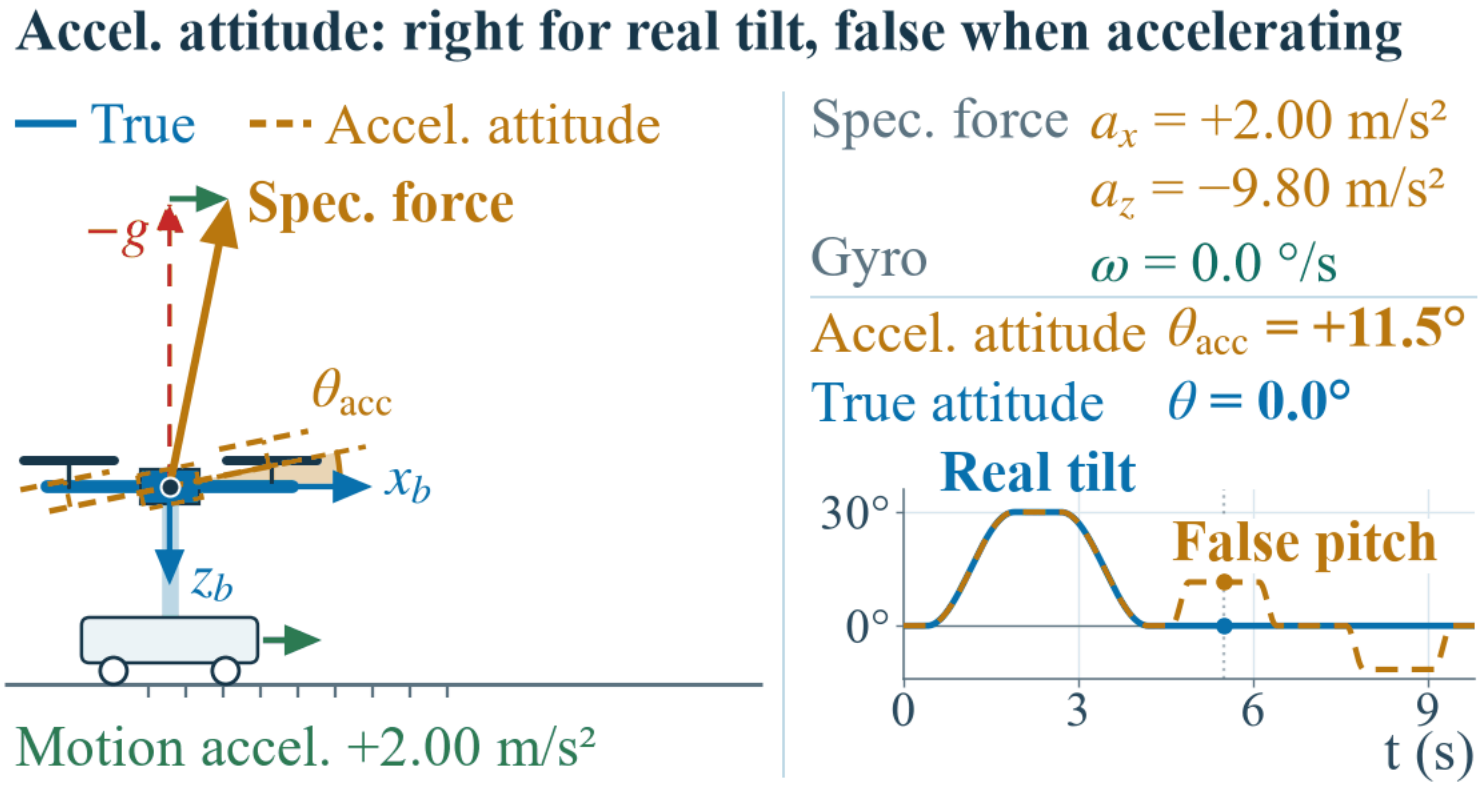


Figure: Accelerometer attitude during tilt and horizontal acceleration (teaching simulation)

True 30° nose-up: Specific force stays vertical as body axes rotate; θ_{acc} and θ both reach 30°. Gyro responds only while rotating. Horizontal acceleration 2 m/s²: $\theta = 0°$, gyro=0, yet specific force tilts forward and $\theta_{acc} \approx 11.5°$; −2 m/s² gives −11.5°. Conclusion: Specific force alone cannot distinguish tilt from horizontal acceleration. Fuse short-term gyro information.

5.2.2 Complementary Filter: Complementary Fusion

Calculate the Fused Angle for One Sample

Given: Previous estimate 10°, angular velocity 2°/s, bias 0, interval 0.1 s, accelerometer attitude 9°, weight 0.9.

$$\theta_{\text{gyro}} = 10^\circ + 2^\circ/\text{s} \times 0.1\,\text{s} = 10.2^\circ$$

$$\theta = \alpha\theta_{\text{gyro}} + (1-\alpha)\theta_{\text{acc}} = 0.9 \times 10.2^\circ + 0.1 \times 9^\circ = 10.08^\circ \quad \text{Eq. 5.5}$$

- The next integration starts from the fused result, 10.08°. Every fusion step corrects subsequent predictions.
- This example uses degrees throughout. If the program uses radians, convert both angles and angular velocities.

5.2.2 Complementary Filter: Time Constant

Weights Change with the Sampling Interval

$$\alpha = \frac{\tau}{\tau + \Delta t}, \qquad 1 - \alpha = \frac{\Delta t}{\tau + \Delta t}$$

Time Constant	Sampling Interval	Gyroscope Prediction Weight
0.3 s	0.02 s	$0.3/0.32 = 0.9375$
0.3 s	0.10 s	$0.3/0.40 = 0.75$
1.5 s	0.02 s	$1.5/1.52 \approx 0.9868$

High-frequency changes come from the gyroscope and low-frequency trends from the accelerometer. τ sets the crossover timescale, in s.

5.2.2 Complementary Filter: Filtering Results

Compare Noise, Drift, and Response on the Same Input

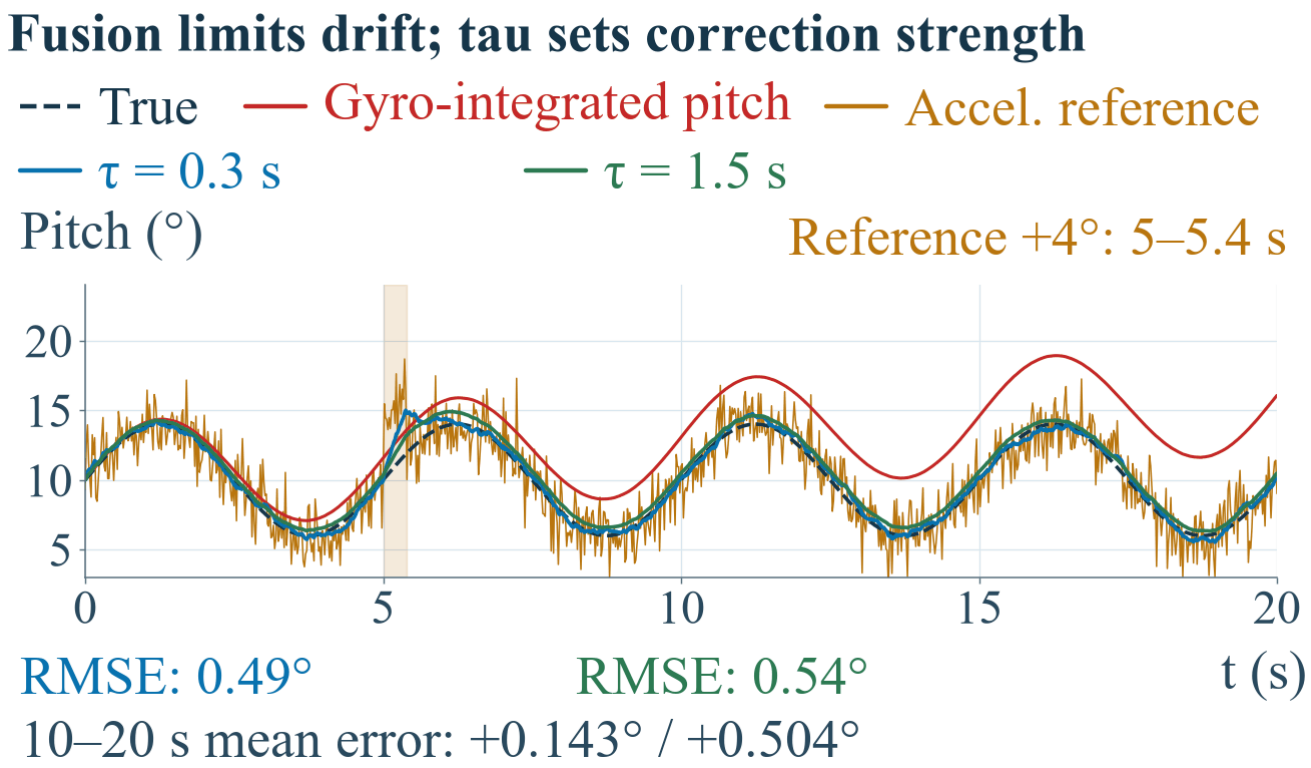


Figure: Complementary attitude: stepwise prediction and correction

Independent teaching calculation, 20 s. Interval: 0.02 s; initial angle: 10°. Residual gyroscope bias: 0.3°/s. Reference rises by 4° at 5–5.4 s. RMSE: Gyroscope 3.48°; complementary, 0.3 s: 0.49°; complementary, 1.5 s: 0.54°.

Conclusion: Red drifts about 6° in 20 s. Both complementary filters reduce RMSE to about 0.5°. The 1.5 s filter responds less to the disturbance at 5 s but retains more bias.

5.2.2 Complementary Filter: Actual Function

Inputs and Limits of the Simplified Attitude Function

e4.1/Attitude_cf.m: function [phi_cf, theta_cf] = Attitude_cf(dt, z, phi_cf_k, theta_cf_k, tao)
z = [gx, gy, gz, ax, ay, az]', in rad/s and m/s²; angles in rad

$$\theta_{\text{am}} = \arcsin(a_x/\|\mathbf{a}\|), \quad \theta_k = \frac{\tau}{\tau + \Delta t}(\theta_{k-1} + g_y \Delta t) + \frac{\Delta t}{\tau + \Delta t}\theta_{\text{am}}$$

- g_y (gy in code) is the body y-axis gyroscope angular velocity (rad/s); θ_{am} is θ_{acc} (rad).
- Attitude_cf.m inputs: Interval, six-axis measurements, previous roll/pitch, and tao; specific force in m/s².
- Small-attitude, single-axis approximation; the pitch expressions agree over the conventional pitch range.
- Check input validity if specific-force magnitude is too small or attitude approaches a singularity.
- **Calls:** Attitude_estimator.m in e4.3 calls this function; the e4.1 driver implements the same formula inside its loop.

5.2.2 Complementary Filter: e4.1 and e4.2 Examples

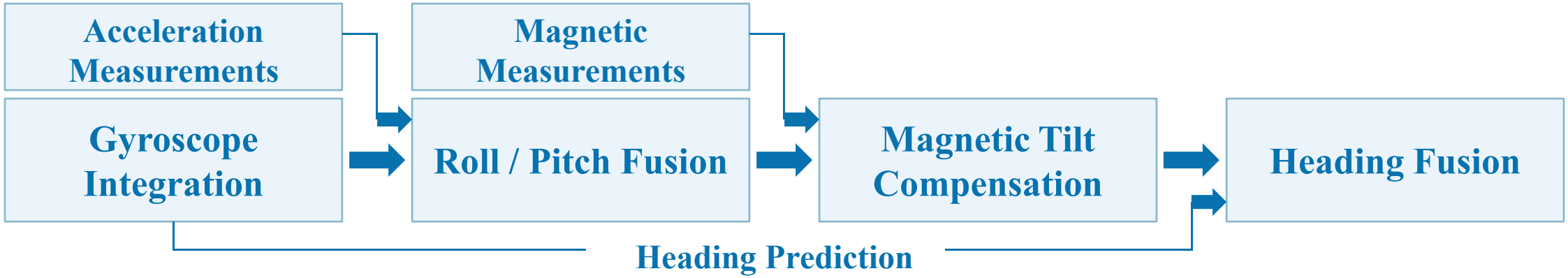
Run Complementary Filtering on Flight Logs and Compare τ

Example folder: [安装目录]/RflySimAPIs/5.RflySimFlyCtrl/1.BasicExps/e4-FilterDesign/
Entries: Readme.pdf (pages 2–6); e4.1/Attitude_estimator0_fly.m; e4.2/Attitude_cf_tao.m

- **Run:** In MATLAB, open e4.1, confirm logdata.mat is there, and run Attitude_estimator0_fly.m; then run Attitude_cf_tao.m in e4.2.
- **Expected:** e4.1 opens pitch/roll figures, gyro/acc/cf above and cf/px4 below; e4.2 opens one pitch figure with τ=0.01, 0.1, 1.
- **Read:** cf is smoother than acc without gyro drift; larger τ approaches gyroscope integration (inferred from $\alpha=\tau/(\tau+\Delta t)$).
- With e4_A.bin: Run Attitude_estimator0.m in e4.1 offline; hardware collection is on the "Log Review" slide in Unit 5.6.5.
- **Troubleshoot:** Match filenames; convert Ts from timestamp (μs) to s. Default tao=0.1 differs from the comparison on the same input.

5.2.2 Complementary Filter: Heading and Applicability

Gravity Alone Cannot Determine Yaw



- Gravity constrains roll and pitch; rotation about it leaves the gravity measurement unchanged.
- Tilt-compensated magnetic measurements give a heading reference.
- **Hard iron:** Fixed fields from onboard magnets and currents. Soft iron: Distortion by ferromagnetic materials.
- Hard iron, soft iron, and motor currents alter magnetic measurements; calibrate and check anomalies.
- **Strengths:** Simple, low computation and delay. Limit: Error covariance is not recursively estimated.

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5.2.3 Kalman Filter: State and Uncertainty

Weighting a Prediction and a Measurement That Disagree

Source	Height Mean	Standard Deviation and Variance (−: Prediction)
Model prediction	1.8 m	$\sigma^- = 0.2$ m, $P^- = 0.04$ m^2
Sensor measurement	2.0 m	$\sigma_z = 0.1$ m, $R = 0.01$ m^2

- The Kalman filter (KF) fuses model and measurement using their uncertainties.
- Variance describes the width of an error distribution, not the error in this sample.
- Here prediction error and measurement noise are uncorrelated; the measurement with smaller variance receives more weight.

5.2.3 Kalman Filter: State and Measurement Equations

Models for State Evolution and Sensor Output

State equation: Control inputs drive the state; process noise captures unexplained changes

$$\mathbf{x}_k = \mathbf{A}_d\mathbf{x}_{k-1} + \mathbf{B}_d\mathbf{u}_{k-1} + \mathbf{w}_{k-1} \quad \text{Eq. 5.6}$$

Measurement equation: Sensors observe combinations of states

$$\mathbf{z}_k = \mathbf{H}_d\mathbf{x}_k + \mathbf{v}_k \quad \text{Eq. 5.7}$$

- $\mathbf{A}_d$: state transition matrix; $\mathbf{B}_d$: input matrix; $\mathbf{H}_d$: measurement matrix.
- $\mathbf{w}$: process noise; $\mathbf{v}$: measurement noise, usually zero-mean and mutually uncorrelated.
- $\mathbf{Q}_d$ and $\mathbf{R}_d$ are their noise covariances.
- $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{z}$ are the state, input, and measurement.

5.2.3 Kalman Filter: Height and Velocity States

Each Matrix Row Represents a Physical Quantity with Units

$$\mathbf{x} = \begin{bmatrix} h \\ v \end{bmatrix}, \quad \mathbf{A}_d = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B}_d = \begin{bmatrix} \Delta t^2/2 \\ \Delta t \end{bmatrix}, \quad \mathbf{H}_d = [1 \quad 0]$$

Quantity	Dimensions	Units and Meaning
State	2×1	Height in m, velocity in m/s
Input and measurement	1×1	Acceleration input in m/s², height measurement in m
Covariance	2×2	Diagonal: m² and (m/s)²; off-diagonal: m²/s

Height is positive upward here. The textbook's NED down position satisfies $h = -z_D$; the acceleration input must also change sign.

5.2.3 Kalman Filter: One-Step Two-State Prediction

One Prediction Step Advances Both Height and Velocity

Given: Height 1.79 m, velocity 0.10 m/s, acceleration input 0.2 m/s², interval 0.1 s; covariance diagonal entries 0.03 m² and 0.01 (m/s)²; no process noise added in this step.

$$h^- = h + v\Delta t + \frac{1}{2}a\Delta t^2 = 1.79 + 0.10 \times 0.1 + \frac{1}{2} \times 0.2 \times 0.1^2 = 1.801 \text{ m}$$

$$v^- = v + a\Delta t = 0.10 + 0.2 \times 0.1 = 0.12 \text{ m/s}$$

$$\mathbf{P}^- = \mathbf{A}_d \mathbf{P} \mathbf{A}_d^\top = \begin{bmatrix} 0.0301 & 0.001 \\ 0.001 & 0.01 \end{bmatrix}$$

Interpretation: With a=0, the height row gives 1.80 m, matching the scalar example; the off-diagonal entry 0.001 m²/s lets a height measurement also correct velocity.

5.2.3 Kalman Filter: Prediction

Propagate the State and Its Covariance Together

State prediction: Use the previous update and known input

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{A}_d \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_d \mathbf{u}_{k-1} \quad \text{Eq. 5.8}$$

Covariance prediction: Propagate existing errors and add process noise

$$\mathbf{P}_{k|k-1} = \mathbf{A}_d \mathbf{P}_{k-1|k-1} \mathbf{A}_d^{\top} + \mathbf{Q}_d \quad \text{Eq. 5.9}$$

- $k|k-1$: time k using measurements only through k−1; abbreviated by superscript "−", with "+" for the update.
- The numerical example retains height only, with known velocity as input (the two-state height row at a=0).
- Previous height 1.79 m, velocity 0.10 m/s, interval 0.1 s: predicted height 1.80 m.

5.2.3 Kalman Filter: Innovation and Gain

The Gain Sets the Measurement Correction

$$\boldsymbol{\nu}_k = \mathbf{z}_k - \mathbf{H}_d \hat{\mathbf{x}}_{k|k-1}$$

$$\mathbf{S}_k = \mathbf{H}_d \mathbf{P}_{k|k-1} \mathbf{H}_d^\top + \mathbf{R}_d$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_d^\top \mathbf{S}_k^{-1}$$

Eq. 5.10

- **Innovation:** Measurement minus the model's predicted measurement.
- **Innovation covariance:** Combined prediction and measurement uncertainty in measurement space.
- Map into measurement space before comparing, so dimensions and units agree.

5.2.3 Kalman Filter: State and Covariance Update

Correct the Mean and Update Uncertainty

State update: Prediction plus weighted innovation

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}_d\hat{\mathbf{x}}_{k|k-1})$$

Eq. 5.11

Covariance update: Measurements reduce uncertainty in constrained directions

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_d)\mathbf{P}_{k|k-1}$$

Eq. 5.12

- $\mathbf{I}$ is the identity matrix. The updated mean and covariance enter the next prediction together.
- Rejecting abnormal measurements (outliers) requires an additional threshold or robust treatment.

5.2.3 Kalman Filter: Scalar Update

One Gain Calculation Gives an Estimate of 1.96 m

Given: Prediction 1.8 m, measurement 2.0 m, predicted variance 0.04 m², measurement variance 0.01 m², direct height measurement.

$$K = \frac{0.04}{0.04 + 0.01} = 0.8, \quad \nu = 2.0 - 1.8 = 0.2\ \text{m}$$

$$\hat{h}^{+} = 1.8 + 0.8 \times 0.2 = 1.96\ \text{m}$$

$$P^{+} = (1 - 0.8) \times 0.04 = 0.008\ \text{m}^2, \quad \sigma^{+} = \sqrt{0.008} = 0.0894\ \text{m}$$

Smaller measurement error variance pulls the update closer to 2.0 m. The fused standard deviation is smaller than that of either input.

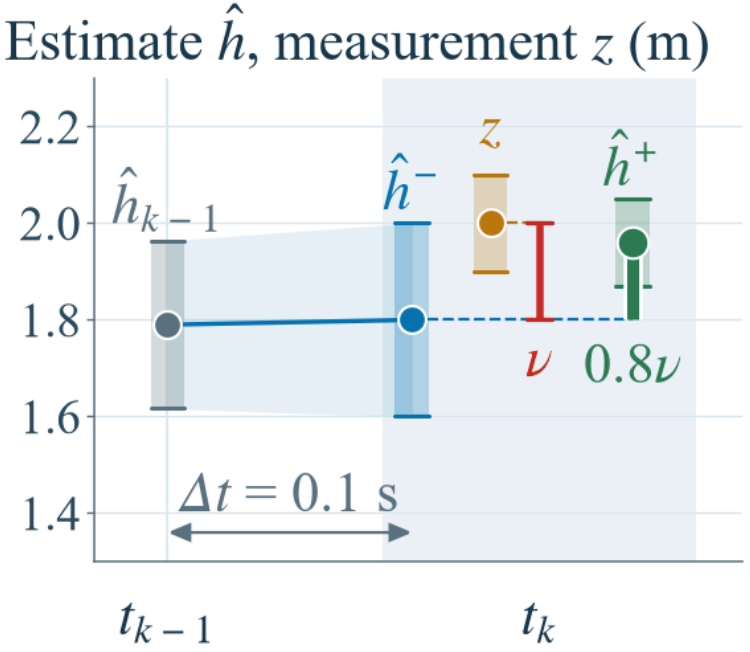
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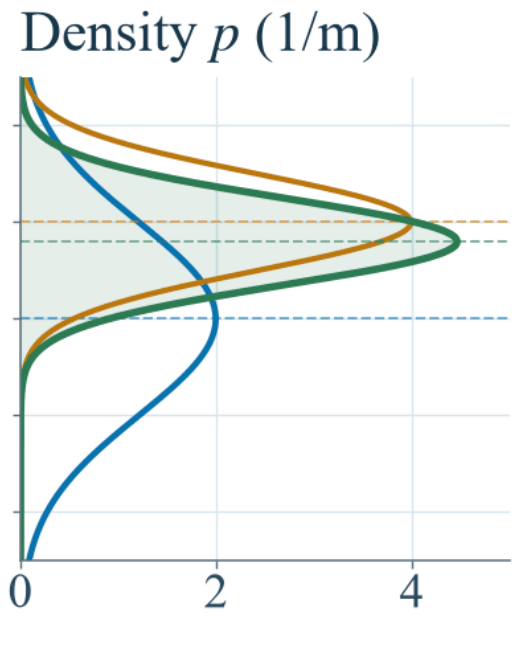
How Fusion Changes the Distribution

Scalar KF: predict and update

Prediction — Measurement — Update

Result: the estimate moves only 0.8ν to 1.96 m; the update is narrower than both inputs





$\hat{h}^- = 1.79 + 0.10 \times 0.1 = 1.80$ m

$P^- = 0.03 + 0.01 = 0.04$ m²

$z = 2.00$ m, $R = 0.01$ m²

$\nu = z - \hat{h}^- = 0.20$ m

$K = 0.04 / (0.04 + 0.01) = 0.8$

0.2 meas. weight $K = 0.8$

$\hat{h}^+ = 1.80 + 0.8 \times 0.20 = 1.96$ m

$P^+ = (1 - 0.8) \times 0.04 = 0.008$ m²

Figure: Height KF teaching step: left, estimate $\pm 1\sigma$; center, same-axis density; right, values

5.2.3 Kalman Filter: Recursive Filtering

Repeat Prediction and Update for 12 s

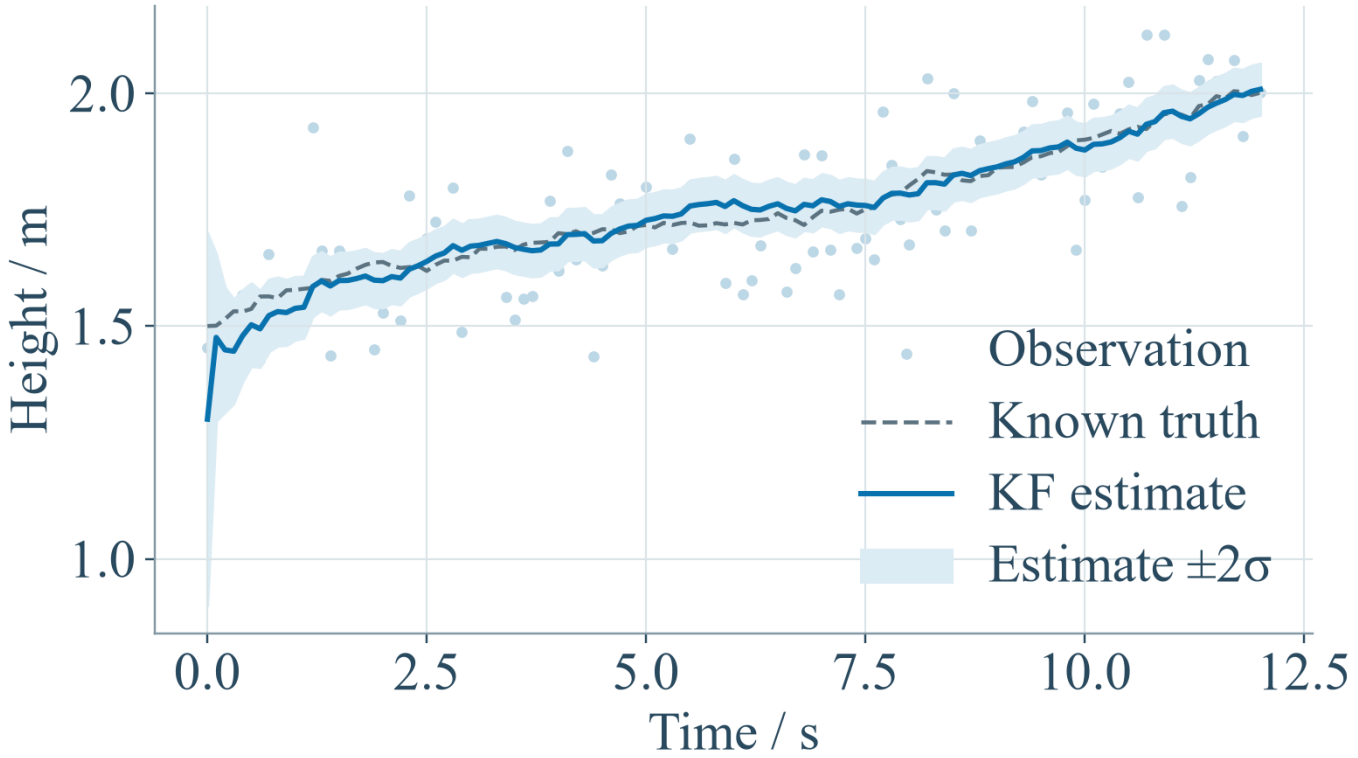


Figure: One-dimensional height recursion for teaching

Independent height teaching calculation.

Interval: 0.1 s; velocity: 0.05 m/s.

$$Q = (0.008\ \text{m})^2$$

$$R = (0.1\ \text{m})^2$$

Initial estimate: 1.3 m; $P_0 = 0.04\ \text{m}^2$

Measurement RMSE: 0.105 m

Estimate RMSE: 0.030 m

Conclusion: The estimate quickly approaches truth from 1.3 m as the $\pm 2\sigma$ band narrows. Its RMSE is about 0.3 times the measurement RMSE.

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5.2.4 Extended Kalman Filter: Nonlinear Extension

Functions Describe Nonlinear States and Measurements

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}) + \mathbf{w}_{k-1} \quad \text{Eq. 5.13}$$

$$\mathbf{z}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k \quad \text{Eq. 5.14}$$

- The extended Kalman filter (EKF) retains prediction and update, with the mean passed through nonlinear functions.
- Attitude changes, coordinate rotations, and nonlinear sensor relations can make mappings depend on the state.
- This unit uses the dimensionless measurement $z = x^2 + v$, choosing the positive branch first.

5.2.4 Extended Kalman Filter: Local Linearization

The Jacobian Gives the Slope Near the Current Estimate

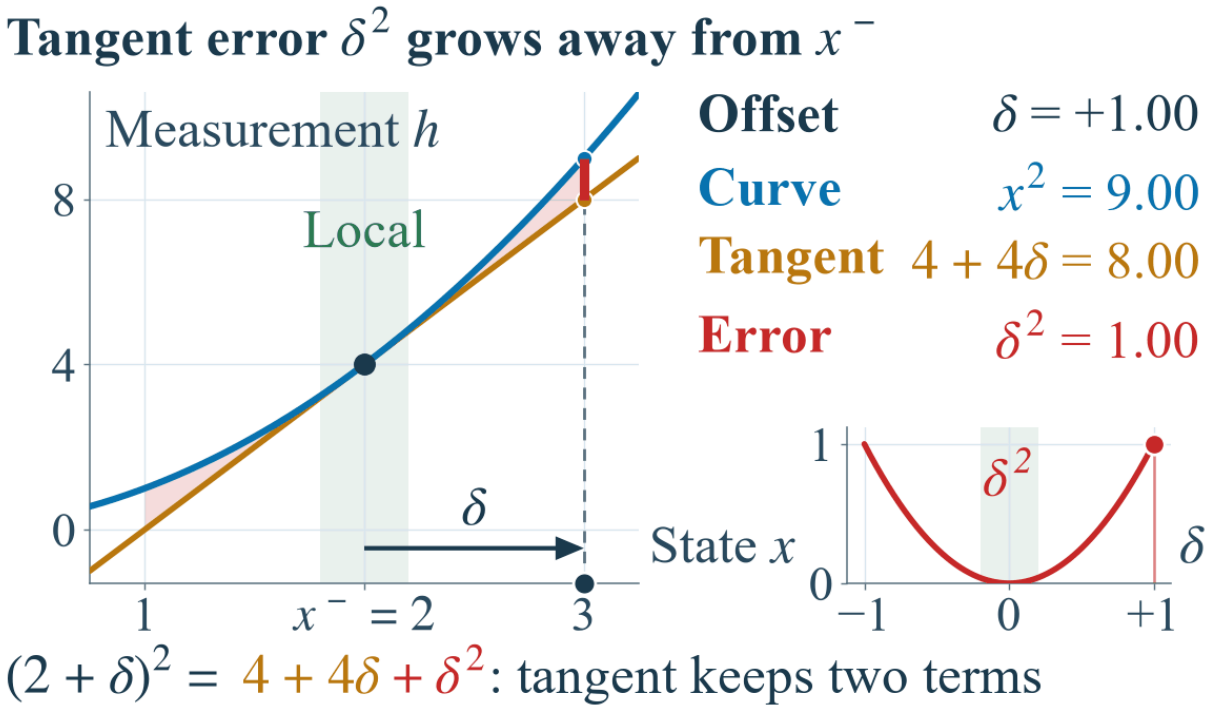


Figure: Local tangent and error δ^2 for a squared measurement (schematic)

The Jacobian is the matrix of partial derivatives of each output with respect to each state.

Here the measurement is $h(x) = x^2$ and the current prediction is $x^- = 2$.

$$H = \left.\frac{\mathrm{d}h}{\mathrm{d}x}\right|_2 = 2 \times 2 = 4$$

The red vertical bar is tangent error δ^2.

The tangent approximates changes only locally.

Recalculate the Jacobian at every step.

5.2.4 Extended Kalman Filter: Jacobians and Prediction

Functions Propagate the Mean; Local Derivatives Propagate Covariance

$$\mathbf{F}_{k-1} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_{k-1}}, \quad \mathbf{H}_k = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}_{k|k-1}} \tag{Eq. 5.15}$$

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{f}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{u}_{k-1}) \tag{Eq. 5.16}$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}_{k-1}^{\top} + \mathbf{Q}_d \tag{Eq. 5.17}$$

- Evaluate **F** at the previous updated state and **H** at the current predicted state.
- Chapter 4's linearization at a fixed operating point becomes local linearization that moves with each recursive estimate.

5.2.4 Extended Kalman Filter: EKF Update

The Innovation Uses the Nonlinear Predicted Measurement

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^\top(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^\top + \mathbf{R}_d)^{-1} \quad \text{Eq. 5.18}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{h}(\hat{\mathbf{x}}_{k|k-1})) \quad \text{Eq. 5.19}$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k|k-1} \quad \text{Eq. 5.20}$$

- Form innovation by subtracting the nonlinear predicted measurement, not the Jacobian times the state.
- A squared measurement has positive and negative branches; the EKF keeps a Gaussian approximation near the current branch.

5.2.4 Extended Kalman Filter: One Nonlinear Step

One Squared-Measurement Update Gives 2.08

Dimensionless teaching example: $x^- = 2$, $P^- = 0.04$, $R = 0.16$, measurement $z = 4.4$, $h(x) = x^2$.

$$H = 4, \quad \nu = 4.4 - 2^2 = 0.4, \quad S = 4^2 \times 0.04 + 0.16 = 0.8$$

$$K = \frac{0.04 \times 4}{0.8} = 0.2, \quad x^+ = 2 + 0.2 \times 0.4 = 2.08$$

$$P^+ = (1 - 0.2 \times 4) \times 0.04 = 0.008$$

Starting from −2 gives −2.08 for the same measurement. A squared measurement cannot distinguish signs; initialization and other information determine the branch.

5.2.4 Extended Kalman Filter: Full Recursion

Check the Full Estimate on the Positive Branch

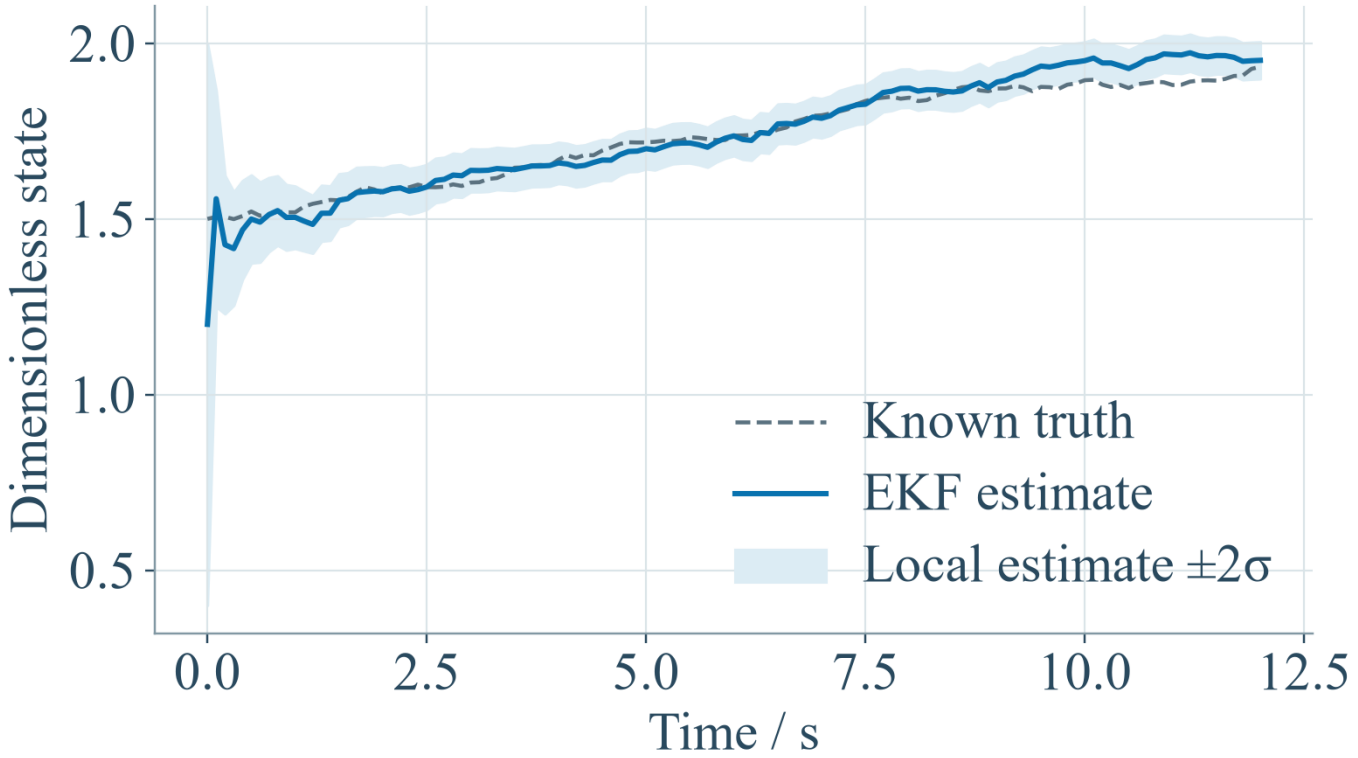


Figure: Nonlinear state recursion on the positive branch for teaching

Separate continuous run; initial estimate 1.2.
State increases by 0.004 per step.
Measurement: $z = x^2 + v$.
Interval: 0.1 s; duration: 12 s.
$Q = 0.007^2$, $R = 0.16$,
$P_0 = 0.16$.
Overall RMSE: 0.045.
Final estimate: 1.951; truth: 1.933.

Conclusion: The estimate quickly approaches truth;
the final difference is about 0.02.
A negative start may select the negative branch.

5.2.4 Extended Kalman Filter: Attitude EKF State

The Actual Example Estimates Bias and a Direction Vector

Example folder: [安装目录]/RflySimAPIs/5.RflySimFlyCtrl/1.BasicExps/e4-FilterDesign/e4.3/
Function: Attitude_ekf.m; see e4-FilterDesign/Readme.pdf, pages 7–9

$$\mathbf{x} = [b_x, b_y, b_z, k_x, k_y, k_z]^\top, \quad \mathbf{P} \in \mathbb{R}^{6\times 6}$$

- First three states: Gyroscope zero-rate biases, rad/s.
- Last three: Gravity-direction components on body x/y/z axes, dimensionless.
- Specific-force measurement: $\mathbf{a}_m \approx -g\mathbf{k}$, $g = \|\mathbf{a}_m\|$.
- At a quasi-static, level attitude, $\mathbf{k} \approx [0,0,1]^\top$.
- PHI approximates the state transition Jacobian in code, with dimensions 6×6.

- Normalize $\mathbf{k}$ after each update: Roll $\phi = atan2(k_y, k_z)$; pitch $\theta = -\arcsin k_x$.

5.2.4 Extended Kalman Filter: Code and Log Comparison

Check Data Conditions Before Comparing Filter Results

Check	Location in the Example	Effect on Results
Time and units	Convert timestamp to seconds; z contains six-axis measurements	Time misalignment changes integration and innovation
Initialization and noise	As-shipped script preset: $\mathbf{k}_0 = [0,0,-1]^\top$ Also check covariance and noise parameters q, r	Incorrect initialization or too little covariance affects correction
Result comparison	theta_cf, theta_ekf, theta_px4	PX4 attitude is a reference estimate, not truth

- Level, stationary forward-right-down model reference: $\mathbf{a}_m \approx [0,0,-g]^\top$, $\mathbf{k}_0 \approx [0,0,1]^\top$.
- Initialization check: See the "e4.3 Example" slide.
- Approximation: Using specific-force magnitude as gravity during maneuvers also affects the measurement model.

5.2.4 Extended Kalman Filter: e4.3 Example

Run e4.3 and Check the Initial Direction

Example folder: [安装目录]/RflySimAPIs/5.RflySimFlyCtrl/1.BasicExps/e4-FilterDesign/e4.3/
Entry: Readme.pdf (pages 7–9); run Attitude_estimator.m

- Run: In MATLAB, switch to e4.3, confirm logdata.mat, Attitude_cf.m, and Attitude_ekf.m are there, then run Attitude_estimator.m.
- Expected: Return to the prompt and open 4 figures: For each of pitch and roll, one cf/ekf/px4 comparison and one am/gm/ekf comparison.
- As-shipped script settings: tao=0.3; w=[0.08, 0.01] passed to q, v=50 passed to r; $\mathbf{P}_0 = 100\mathbf{I}$, $\mathbf{k}_0 = [0,0,-1]^\top$.

Average specific force from the first quasi-static segment → **Check $\mathbf{a}_m + g\mathbf{k}_0 \approx \mathbf{0}$** → **If not satisfied, check stationarity and mounting, axis transforms and units** → **Reset initialization using the measurement model, then inspect innovation**

Chapter Outline

Sensor Calibration, Filtering, and State Estimation

Chapter 5
Filtering, State Estimation,
and Low-Level Control

5.1 Filtering and Low-Level Control

5.2 Sensor Calibration and Estimation

5.3 Controller Design and Allocation

5.4 Control Performance and Tuning

5.5 PX4 Architecture and Interfaces

5.6 Control Development and Testing

5.7 Other Vehicles and Advanced Cases

5.8 Exercises and Classroom Discussion

5 Units in This Section

5.2.1 Sensor Calibration and Measurement Model
Textbook: Sec. 5.1.3.2, Part 1; Sec. 5.3.1

5.2.2 Complementary Filter
Textbook: Sec. 5.1.3.2, Part 2; Sec. 5.3.1

5.2.3 Kalman Filter
Textbook: Sec. 5.1.3.2, Part 3

5.2.4 Extended Kalman Filter
Textbook: Sec. 5.1.3.2, Part 4; Sec. 5.3.1

5.2.5 Filter Comparison and Validation
Textbook: Sec. 5.1.3.2, Part 5

5.2.5 Filter Comparison and Validation: Signal Filtering

Choose Signal Filters for the Noise and Statistical Conditions

Method	Basic Mechanism	Conditions and Cost
Median filter	Sort a window and select the median	Suppresses spikes; the window can remove brief details
Wiener filter	Minimum mean-square error under a statistical model	Requires signal/noise statistics; commonly assumes linearity and stationarity
Adaptive filter	Update filter coefficients from data	Least mean squares (LMS) may converge slowly; recursive least squares (RLS) has high computational cost

- Choose by noise type (spikes or broadband), knowledge of statistics, and allowable delay.
- A median filter needs no motion model and does not output state covariance.

5.2.5 Filter Comparison and Validation: Nonlinear State Filtering

Distribution Propagation Determines Applicability

Method	State Distribution Propagation	Limits
EKF	Nonlinear functions propagate the mean; derivatives propagate covariance	Local linearization; sensitive to initialization
Unscented Kalman filter (UKF)	Map a prescribed set of samples (sigma points) through nonlinear functions; reconstruct mean and covariance	Avoids explicit derivatives; accuracy still depends on the model and parameters
Particle filter (PF)	Approximate the posterior with many weighted samples	Can represent multiple modes; requires management of particle degeneracy and computation

Greater complexity does not guarantee greater accuracy. State dimension, available measurements, initial distribution, and the real-time computing budget determine the choice.

5.2.5 Filter Comparison and Validation: Result Evaluation

Does a Smoother Estimate Improve Control Feedback?

$$\mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(\hat{x}_k - x_k)^2}$$

Known Truth	No Truth Available	Real-Time Implementation
Evidence: RMSE, bias, phase lag or time delay **Meaning:** Both error and delay affect control	**Evidence:** Innovation, independent measurements, time alignment **Meaning:** Consistency checks alone cannot establish accuracy	**Evidence:** Computation per step, update rate, data age **Meaning:** Computation slower than sampling changes closed-loop conditions

N: sample count; $\hat{x}_k$: estimate; x_k: truth. RMSE has the same units as the state.

5.2.5 Filter Comparison and Validation: Evaluation Example

A Smoother Estimate May Have Greater Error

Data: "Compare Noise, Drift, and Response on the Same Input" teaching example; duration 20 s, interval 0.02 s, residual gyroscope bias 0.3°/s.

Complementary, τ=0.3 s	Complementary, τ=1.5 s
Root mean square of consecutive changes: 0.122°	**Root mean square of consecutive changes:** 0.075°
Overall RMSE: 0.49°	**Overall RMSE:** 0.54°
Mean error over 10–20 s: 0.14°	**Mean error over 10–20 s:** 0.50°

$$e_{ss} = b_g\tau: \quad 0.3 \times 0.3 = 0.09°, \quad 0.3 \times 1.5 = 0.45°$$

Conclusion: Larger τ makes the curve smoother but leaves a larger steady-state error from gyroscope bias. Evaluate error, bias, and delay together.

5.2.5 Filter Comparison and Validation: Reliability Limits

Locate the Failed Information Stage When Anomalies Occur

Symptom	Evidence	Response
Aggressive maneuver	Specific-force magnitude and innovation keep increasing	Reduce confidence in gravity measurements or improve the model
Timestamp misalignment	The same event appears at different times	Align data before fusion
Measurement interruption	Prediction only; uncertainty depends on the model	Adapt feedback and degradation to available states
Incorrect initialization	Wrong branch or persistently large innovation	Check initialization, coordinates, and startup conditions

Confidence and validity flags guide feedback selection. Estimation accuracy usually declines during prolonged measurement loss.

Classroom Exercise

Which Way Does the Estimate Move as Measurement Variance Increases?

Exercise: Keep prediction 1.8 m, predicted variance 0.04 m², and measurement 2.0 m. Change measurement variance to 0.16 m². Find the gain, estimate, and updated variance.

$$K = \frac{0.04}{0.04 + 0.16} = 0.2, \quad \hat{h}^{+} = 1.8 + 0.2(2.0 - 1.8) = 1.84\ \text{m}$$

$$P^{+} = (1 - 0.2) \times 0.04 = 0.032\ \text{m}^2$$

- **Answer guidelines:** Measurement weight falls from 0.8 to 0.2, moving the estimate closer to prediction.
- Standard deviation 0.4 m means variance 0.16 m²; incorrectly substituting 0.4 gives K≈0.09.

Classroom Exercise

One Calibration Step and One Squared-Measurement Update

Question 1: Calibration compensation. The example corrects readings using $a_{\text{cal}} = K_a(a_{\text{raw}} + b_a)$. Given $K_a = 1.02$, $b_a = 0.20\ \text{m/s}^2$, and raw reading 9.40 m/s² on one axis:
① Find the corrected value. ② In measurement model $y = sx + b$, find s and b, and explain the sign relation between b_a and b.

Question 2: One EKF update. Measurement function $h(x) = x^2$ (dimensionless), prediction $x^- = 3$, $P^- = 0.05$, $R = 0.25$, and measurement $z = 9.6$.
Find H, innovation ν, S, K, x^+, and P^+. Explain why innovation cannot be calculated using Hx^-.

Answer Guidelines

Establish the Signs and Measurement Function Before Substitution

Question 1: Corrected value $1.02\times(9.40+0.20)\approx 9.79$ m/s²; incorrectly using -0.20 gives 9.38 m/s²; $s = 1/K_a \approx 0.980$, $b = -b_a = -0.20\ \mathrm{m/s^2}$. Model bias and compensation have opposite signs.

$$H = 2x^- = 6, \quad \nu = 9.6 - 3^2 = 0.6, \quad S = 6^2 \times 0.05 + 0.25 = 2.05$$

$$K = \frac{0.05 \times 6}{2.05} \approx 0.146, \quad x^+ = 3 + 0.146 \times 0.6 \approx 3.088$$

$$P^+ = (1 - KH)P^- = \frac{0.25}{2.05} \times 0.05 \approx 0.0061$$

- Question 2 pitfall: Using $Hx^- = 18$ in innovation gives $\nu = -8.4$ and $x^+ \approx 1.77$, reversing the correction direction.
- For complementary weights and method selection, see Deck 5.8, Exercises 2 and 7.

Section Summary

Usable State Feedback Requires Valid Models and Measurements

Method	Problem Addressed	Common Pitfalls
Calibration	Relating readings to physical quantities	Compensation sign, scale factors, axis directions
Complementary filtering	Fusing short-term changes with a long-term reference	Maneuvering specific force, sampling interval, weights
KF / EKF	Recursive estimation and uncertainty propagation	Noise variances, innovation space, local initialization
Method selection	Choose and validate by error, delay, and computing cost	Judging only smoothness; treating PX4 attitude as truth

- Pass to the controller: State values, units, coordinates, timestamps, and confidence.
- Deck 5.3 uses targets and these states to form errors, then calculates achievable control actions.