Chapter Outline

Trim, Linearization, and Dynamic Systems

Chapter 4
Vehicle Motion Modeling and Simulation

4.1 Modeling Fundamentals and Kinematics
4.2 Rigid-Body Dynamics and Solvers
4.3 Propulsion, Forces, and Observations
4.4 Trim, Linearization, and Dynamics
4.5 Multicopter Modeling and Simulation
4.6 Measurement and Static Identification
4.7 Dynamic Identification and Refinement
4.8 Typical Vehicles and Extensions
4.9 Modeling Tools and Deployment
4.10 API and SDK Modeling Interfaces
4.11 Examples and Integrated Practice
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5 Units in This Section

4.4.1 Operating Points and Trim

Textbook: Sec. 4.1.4.5

4.4.2 Local Linearization and States

Textbook: Sec. 4.1.4.5

4.4.3 State Space and Transfer Functions

Textbook: Secs. 4.1.4.3, 4.1.4.5

4.4.4 Step and Frequency Responses

Read responses using time constant, magnitude ratio, and phase

4.4.5 Sampling, Hold, and Discretization

Textbook: Sec. 4.1.4.5

4.4.1 Operating Points and Trim: Separate Inputs from Environment

Distinguish Control Inputs from Environmental Forces

- Controller-adjusted effects are controls; gravity, wind, and ground constraints are usually environmental. Actuation and model boundaries, not force names, determine the class.
- Ideal quadcopter: treat total thrust and three torques as independent virtual controls, $u = [f, \tau_x, \tau_y, \tau_z]^T$, then use an allocator to obtain four rotor commands.
- Planar ground vehicle: with only longitudinal drive and yaw torque modeled, $u = [f, \tau_z]^T$. Lateral tire–ground forces follow from constraints and tire states.
- Boundaries: tire–ground friction can provide propulsion or resistance. A controller moves fixed-wing control surfaces, but actual aerodynamic forces also depend on airspeed and angle of attack. State each modeling choice.

Define controllable inputs and environmental effects before writing operating-point force balance.

4.4.1 Operating Points and Trim: Hover Force and Torque Balance

Hover Starts with Force and Torque Balance

- Write the nonlinear model as $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$. At hover operating point $(\mathbf{x}_0, \mathbf{u}_0)$, state derivatives vanish; position and yaw may be specified by the task.
- For four identical level rotors and no other external forces, total thrust equals weight and net body torque $\mathbf{M}_{b0}$ is zero. Allocation maps four controls to four rotor inputs.
- Motors keep spinning, so physical input is nonzero. Zero perturbation input means maintaining the operating point.

$$\mathbf{x}_0 = [x_0, y_0, z_0, 0,0,0,0,0, \psi_0, 0,0,0]^{\mathrm{T}}, \quad \mathbf{u}_0 = [mg, 0,0,0]^{\mathrm{T}}$$

$$f(\mathbf{x}_0, \mathbf{u}_0) = \mathbf{0}, \quad \sum_i T_{i0} = mg, \quad \mathbf{M}_{b0} = \mathbf{0}$$

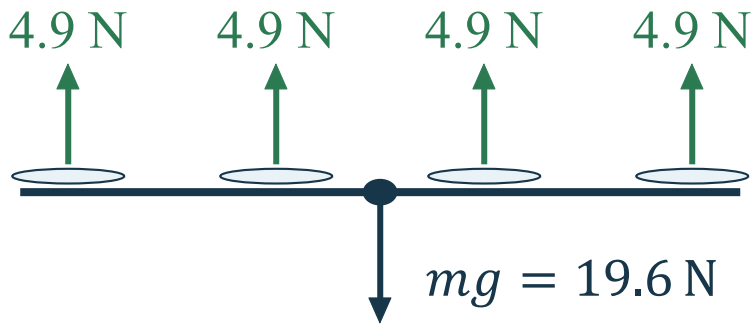
Hover needs nonzero thrust input; net force and net torque balance to zero.

4.4.1 Operating Points and Trim: Thrust per Rotor, Then Speed

Calculate Hover Speed from the Stated Parameters

- Parameters: $m = 2\,\text{kg}$, $g = 9.8\ \text{m/s}^2$; 4 identical rotors; $c_\text{T} = 2.0 \times 10^{-5}\ \text{N/(rad/s)}^2$.
- Each rotor needs $T_0 = mg/4 = 4.9\,\text{N}$. From $T_0 = c_\text{T}\omega_0^2$, obtain $\omega_0 \approx 494.975\,\text{rad/s}$.
- Use nominal values here. Substituting $c_\text{T} \approx 2.657 \times 10^{-5}$, the fitted value to be used in slide section 4.6, gives $\omega_0 \approx 429.4\,\text{rad/s}$.

$$\omega_0 = \sqrt{\frac{mg}{4c_\text{T}}} = \sqrt{245000} \approx 494.975\,\text{rad/s}$$



Equal rotor thrusts: zero net force and torque

Calculate the operating point before examining small changes around it.

4.4.1 Operating Points and Trim: From Hover to Steady Motion

Solve Trim Conditions, Then Check All State Derivatives

- **Substitute back:** $4c_T\omega_0^2 = 19.6\,\mathrm{N} = mg$, so vertical acceleration is zero. Equal rotor speeds give zero three-axis torque and angular acceleration. Motors maintain ω_0 at actuator steady state.
- **Procedure:** specify mass, attitude, and environment; solve thrust and rotor inputs; substitute state and input into the nonlinear equations and check whether velocities and angular rates keep changing.
- **Steady translation differs:** position still changes in level constant-speed flight. Not every derivative of a state containing position can be zero. Define trim for that motion or in a moving reference.
- **Purpose:** trim provides a defined linearization reference. Arbitrary attitude and throttle may immediately accelerate the vehicle; successful initialization does not establish trim.

Trim follows balance conditions for a specified motion, not arbitrary initialization values.

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4.4.3 State Space and Transfer Functions

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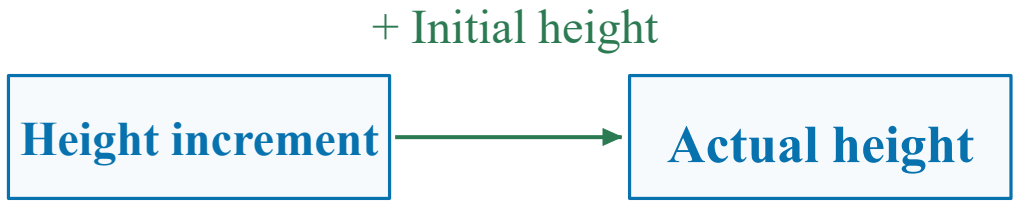
4.4.2 Local Linearization and States: Separate Absolute Values and Increments

Perturbations Measure Deviation from a Reference

- Define $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}_0$, $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}_0$ (written as $\tilde{\mathbf{x}}$, $\tilde{\mathbf{u}}$ in the textbook). From a total thrust of 19.6 N, a 0.4 N increment gives an absolute input of 20.0 N.



- The linear model computes deviations from the operating point. Add initial height to the output height increment. Feeding an increment to an absolute-value interface changes its physical meaning.



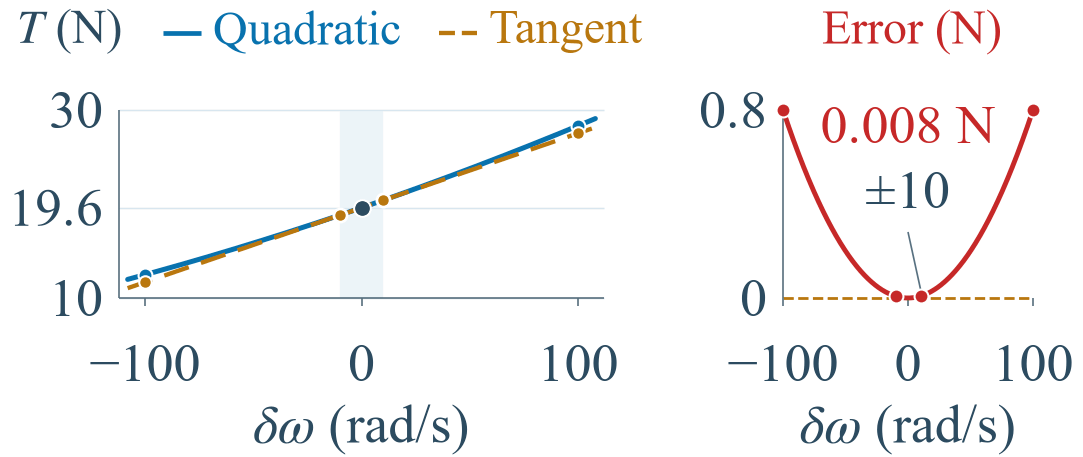
Subtract the reference at model input; add it back at output

Define perturbations about the operating point; handle reference values at both input and output.

4.4.2 Local Linearization and States: Derivatives as Local Sensitivities

First-Order Linearization Keeps Slopes and Drops Higher Orders

- About the operating point, retain only first-order terms in $\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u})$: $\delta\dot{\mathbf{x}} = A\delta\mathbf{x} + B\delta\mathbf{u}$.
- A maps state changes to derivative changes; B maps input changes. The thrust–speed graph illustrates a local approximation with a fixed tangent point.



Fixed tangent: small changes agree; large ones diverge

$$A = \left.\frac{\partial f}{\partial \mathbf{x}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)}, \qquad B = \left.\frac{\partial f}{\partial \mathbf{u}}\right|_{(\mathbf{x}_0,\mathbf{u}_0)}$$

Linear-model coefficients describe sensitivities near the operating point.

4.4.2 Local Linearization and States: A Linear Slope from a Quadratic Law

Expand Total Thrust About the Operating Point

- **Plant:** teaching quadcopter $m = 2\,\text{kg}$, per-rotor $T_i = c_T\omega^2$, $c_T = 2 \times 10^{-5}\,\text{N/(rad/s)}^2$, with all rotors at equal speed.
- **Expansion:** total thrust $T = 4c_T\omega^2$. Substitute $\omega = \omega_0 + \delta\omega$: the constant is hover thrust, the first-order term is local slope, and the squared perturbation is approximation error.
- **Meaning:** at $\omega_0 \approx 494.975\,\text{rad/s}$, a 1 rad/s speed rise gives approximately 0.079196 N more total thrust to first order. This slope belongs to this operating point.

$$\delta T = 8c_T\omega_0\delta\omega + 4c_T(\delta\omega)^2$$

$$\delta T \approx 0.079196\,\delta\omega$$

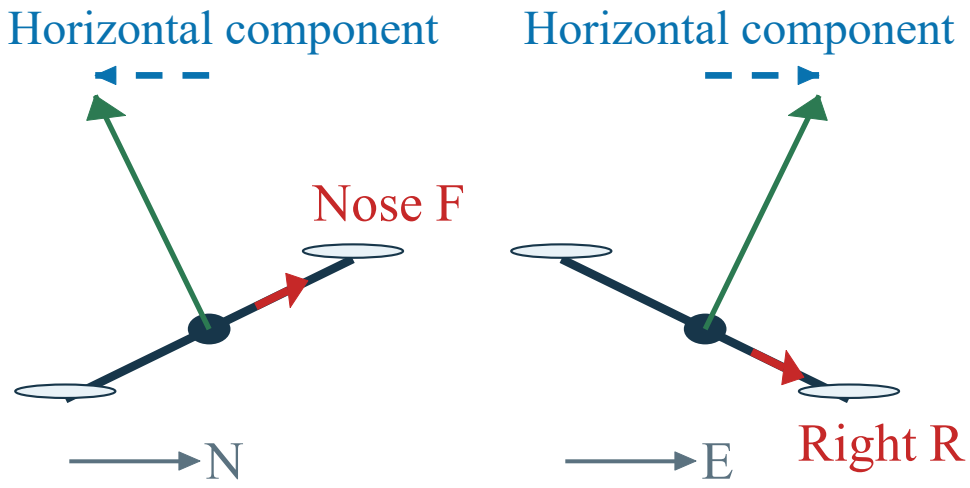
Linearization keeps first-order changes locally; the original nonlinear law still sets the approximation range.

4.4.2 Local Linearization and States: Horizontal and Vertical Coupling Near Hover

How Small Tilt Produces Horizontal Acceleration Near Hover

- Earth north-east-down (NED), body front-right-down (FRD). At level, zero-yaw hover, rotate upward thrust to Earth coordinates. Small angle changes give:
- Positive pitch accelerates negative north; positive roll, positive east. More thrust reduces downward acceleration. Use radians: $1^\circ = 0.01745$.

$$\begin{bmatrix} \delta\dot{v}_N \\ \delta\dot{v}_E \\ \delta\dot{v}_D \end{bmatrix} \approx \begin{bmatrix} -g\delta\theta \\ g\delta\phi \\ -\delta T/m \end{bmatrix}$$



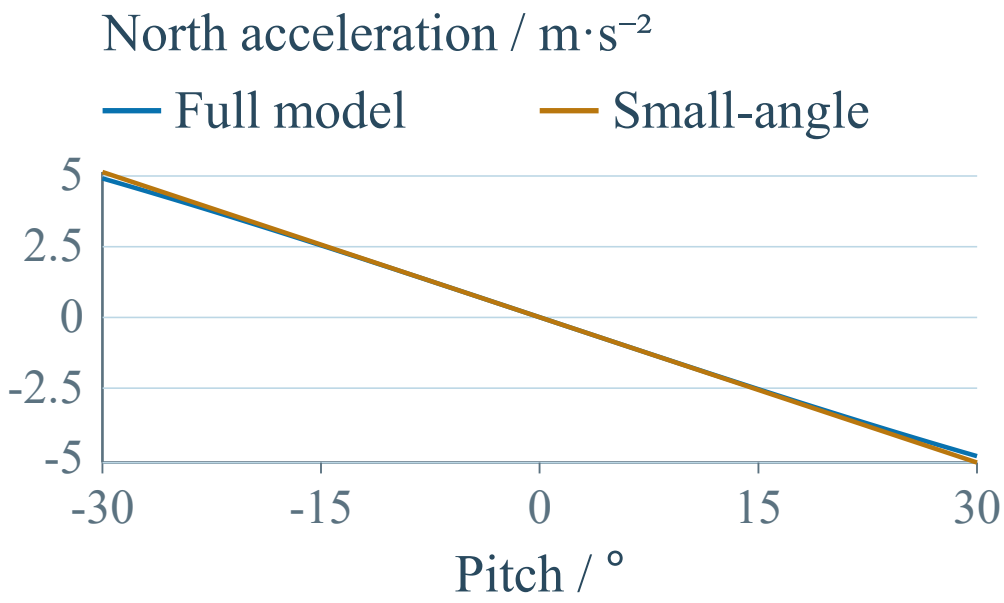
Thrust components (tilt exaggerated)

Near hover, tilt redirects weight-balancing thrust into horizontal acceleration.

4.4.2 Local Linearization and States: Small-Angle Error as Tilt Increases

How Does Approximation Error Change from 5° to 30°?

- With total thrust held at mg and pitch only, north acceleration is $-g\sin\theta$, approximated by $-g\theta$. At 5°, the results are −0.854 and −0.855 m/s².
- At 30°, they are −4.900 and −5.131 m/s². This error comes from omitted higher-order trigonometric terms; reducing the integration step cannot remove it.



Angles displayed in degrees; calculations use radians

Judge local linearization by perturbation range and intended use.

4.4.2 Local Linearization and States: Stored Values and Independent Perturbations

Global Quaternions and a Local 12-State Model

- Nonlinear simulation stores 4 quaternion components; unit norm leaves 3 attitude degrees of freedom. A constant-mass rigid body stores 13 values but has 12 independent states.

- Hover is far from Euler pitch singularities ($\pm 90°$). Locally, position, velocity, small angles, and angular rate each use 3D increments. The quaternion norm adds no independent attitude degree of freedom.

$$\mathbf{x} = [x, y, z, v_x, v_y, v_z, \phi, \theta, \psi, p, q, r]^{\mathrm{T}}$$

	Components	Independent
Position	3	3
Velocity	3	3
Quaternion	4	3
Angular rate	3	3

Total: 13 values → 12 independent states

Stored components differ from independent degrees of freedom

Global attitude retains constraints; the local model uses independent small perturbations.

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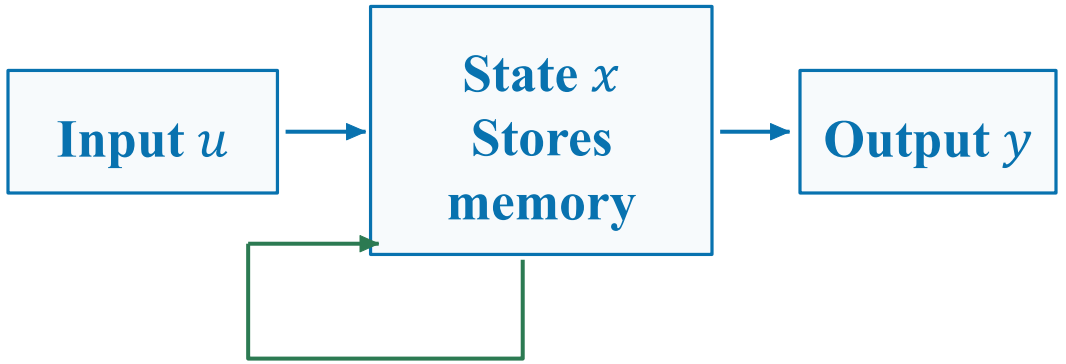
4.4.5 Sampling, Hold, and Discretization

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4.4.3 State Space and Transfer Functions: State Space of a First-Order Motor

State Space Separates Internal Memory from Observed Outputs

- A first-order motor uses current speed $x = \omega$ as its state, with $\dot{x} = -x/\tau + (K/\tau)u$. The same command from different initial states gives different transients.
- Direct speed measurement: $y = x$. Generally, C maps states to outputs; D gives direct input effects. Measurement-channel count need not equal state count.



$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

States store dynamic memory; outputs are observations formed from states and inputs.

4.4.3 State Space and Transfer Functions: From Component Models to State Matrices

Motor and Vertical Motion Combine into Two States

- **States:** choose $\delta x = [\delta v_D, \delta\omega]^{\mathsf{T}}$, where v_D is NED downward velocity and ω common rotor speed. Input δu is the common normalized command increment.
- **Connection:** increased thrust reduces down acceleration, so $\delta\dot{v}_D = -0.039598\delta\omega$. Teaching motor: $\tau = 0.05\,s$, steady-speed gain $K_u = 1000\,\mathrm{rad/s}$ per unit command.
- **Matrix:** row 1 links rotor speed to translation; row 2 describes motor decay, with $1/\tau = 20$, $K_u/\tau = 20000$. Input first drives the motor, then thrust changes velocity.

$$\delta\dot{x} = \begin{bmatrix} 0 & -0.039598 \\ 0 & -20 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ 20000 \end{bmatrix} \delta u$$

Each state-matrix row and column corresponds to a physical causal relationship.

4.4.3 State Space and Transfer Functions: Transfer Functions and Zero Initial Conditions

A Transfer Function Is the Zero-State Output/Input Laplace Ratio

1 Differential equation

- In first-order model $\tau\dot{y} + y = Ku$, u is input, y output; K sets steady gain and τ response speed.

2 Laplace transform

- Laplace input/output: $U(s), Y(s)$, variable s. A derivative becomes $sY(s) - y(0)$; do not drop nonzero initial conditions.

$$\tau[sY(s) - y(0)] + Y(s) = KU(s)$$

The initial term vanishes only for zero initial perturbation

3 Output/input ratio

$$G(s) = Y(s)/U(s) = \frac{K}{\tau s + 1} \qquad (y(0) = 0)$$

$G(s)$ is a transformed-domain ratio, not instantaneous $y(t)/u(t)$.

4.4.3 State Space and Transfer Functions: From State Equations to Input–Output Behavior

The Same Vertical Model as a Cascade Transfer Function

- **Scope:** input δu to output δv_D, with zero initial perturbation. The motor produces a speed increment; thrust changes acceleration; integration gives velocity.
- **Cascade:** $\delta\omega(s) = 1000\,\delta U(s)/(0.05s+1)$, $\delta V_D(s) = -0.039598\,\delta\omega(s)/s$. Multiply to obtain the expression below. The minus sign reflects positive-down NED coordinates.
- **Input check:** a sustained positive increment keeps thrust above weight, so the ideal zero-drag body keeps accelerating. A steady motor speed need not imply a finite steady vehicle velocity.

$$\frac{\delta V_D(s)}{\delta U(s)} = -\frac{39.598}{s(0.05s+1)}$$

Every transfer-function factor should trace back to a physical link in the propulsion chain.

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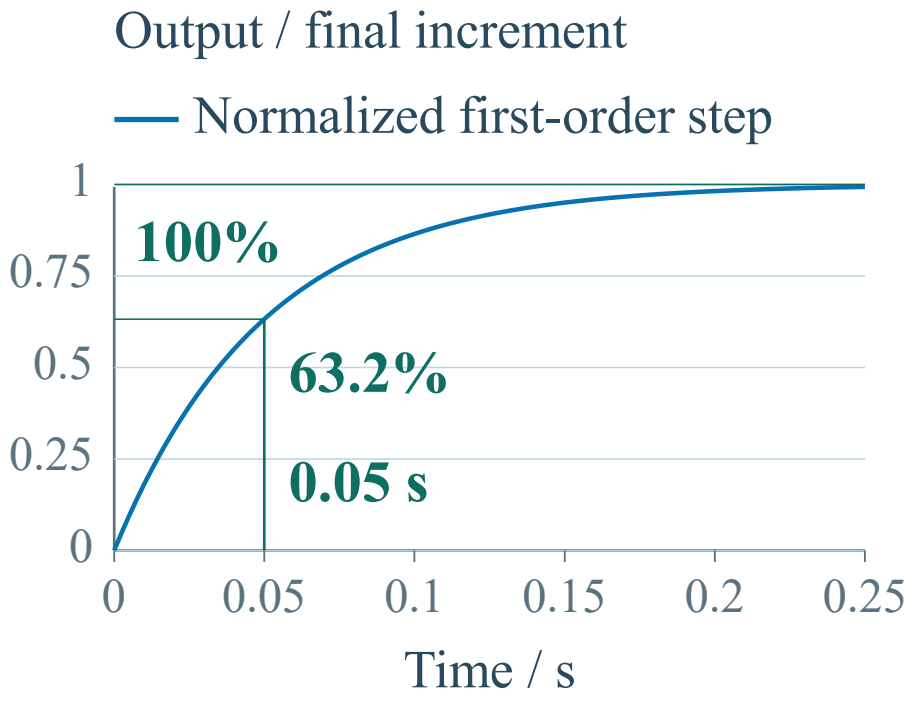
4.4.5 Sampling, Hold, and Discretization

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4.4.4 Step and Frequency Responses: First-Order Step Gain and Time Constant

The Time Constant Marks 63.2% of a Step Change

- A unit step from zero initial state gives $y(t) = K(1 - e^{-t/\tau})$; gain sets the final increment. At $t = \tau$, output reaches 63.2% of that increment.
- For teaching τ=0.05 s: 63.2% at 0.05 s, 95.0% at 0.15 s, and 99.3% at 0.25 s. These ratios require a delay-free first-order structure.
- An initial pause may indicate delay. Clear oscillation or overshoot requires a richer structure than a first-order model without zeros.



Teaching step response, τ=0.05 s

4.4.4 Step and Frequency Responses: Reading Sinusoidal Magnitude and Phase

Magnitude Ratio and Phase Describe Sinusoidal Steady Response

- For input $A\sin(\Omega t)$, a stable linear system gives a same-frequency sinusoid after transients. Its magnitude ratio is $|G(j\Omega)|$ and phase difference $\angle G(j\Omega)$.
- Set $s = j\Omega$ in the transfer function to obtain the selected frequency response, where $j^2 = -1$. Negative phase means output lag; it is not always a fixed time delay.
- The formula below applies to a delay-free first-order link with $K > 0$, $\tau > 0$. For measured magnitude/phase, fix the operating point and input amplitude, and remove transients.

$$|G(j\Omega)| = \frac{K}{\sqrt{1+(\Omega\tau)^2}}, \qquad \angle G(j\Omega) = -\arctan(\Omega\tau)$$

Frequency response describes attenuation and lag at each frequency.

4.4.4 Step and Frequency Responses: Frequency Units and Phase Timing

Convert Hz, rad/s, and Phase-Time Differences

- Frequency f in Hz counts cycles per second; angular frequency $\Omega = 2\pi f$ counts phase radians per second. Convert before evaluating $G(j\Omega)$.
- Example: 2 Hz has period 0.5 s. An output peak 0.05 s later lags by 1/10 cycle, or $-36°$.
- Interpret peak lag at the corresponding frequency and with peaks paired within the same cycle. Mismatching cycles or using degrees in a radian formula gives wrong delays.

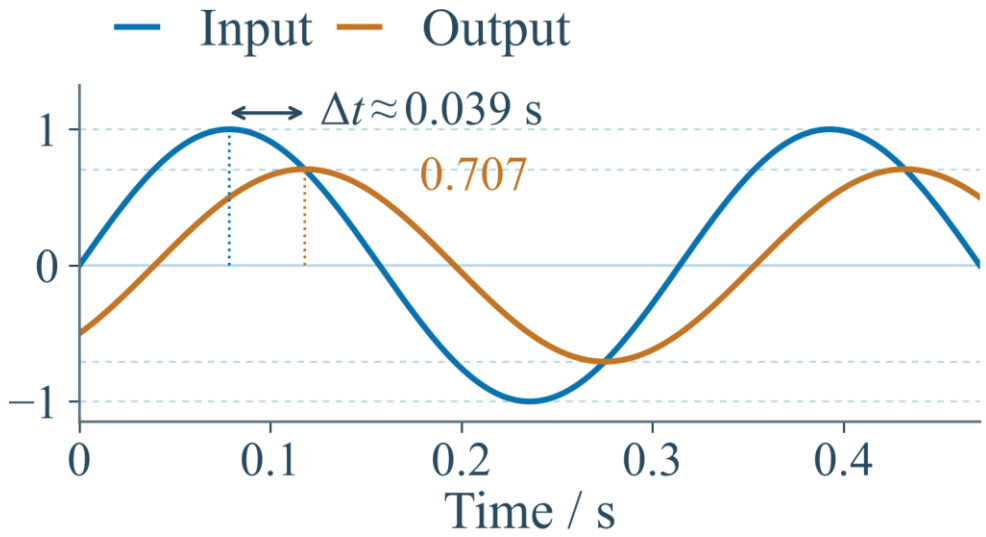
$$\Omega = 2\pi f, \qquad \varphi = -360^\circ f \Delta t$$

Wrong frequency units also give the wrong time lag from a phase difference.

4.4.4 Step and Frequency Responses: Compare Sinusoidal Input and Output

At the Corner: Magnitude 0.707, Lag One-Eighth of a Cycle

- **Example:** $K = 1$, $\tau = 0.05$ s, input 3.183 Hz, so $\Omega = 20$ rad/s, $\Omega\tau = 1$.
- **Magnitude:** after transients, output amplitude is 0.707, matching $1/\sqrt{2}$.
- **Phase:** the output peak is about 0.039 s later, one eighth of the 0.314 s period: $-360°/8 = -45°$.



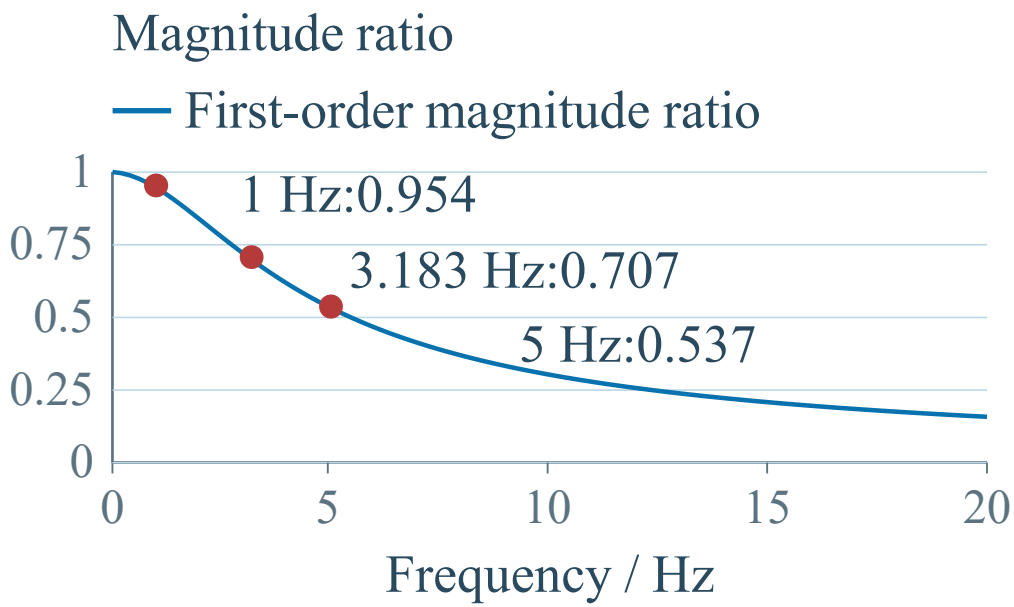
Steady input/output at 3.183 Hz
(teaching example, not platform measurements)

One time constant determines both magnitude attenuation and phase lag.

4.4.4 Step and Frequency Responses: First-Order Magnitude and Phase

How One Time Constant Limits Trackable Frequencies

- For $K = 1, \tau = 0.05$ s: at 1 Hz, magnitude 0.954 and phase $-17.44°$; at 5 Hz, magnitude 0.537 and phase $-57.52°$. Faster changes are harder to follow.
- At $\Omega\tau = 1$, magnitude is $1/\sqrt{2}$ and phase $-45°$, here at 3.183 Hz. A decibel plot uses $20\log_{10}|G|$ and must label its scale.



Delay-free first-order magnitude ratio; not decibels

As frequency rises, finite response time reduces output magnitude and increases lag.

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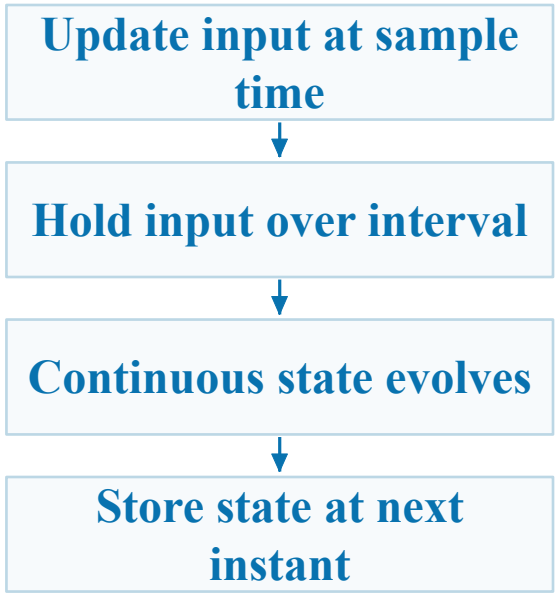
4.4.5 Sampling, Hold, and Discretization

Textbook: Sec. 4.1.4.5

4.4.5 Sampling, Hold, and Discretization: Sampling and Zero-Order Hold

Sampling Sets Instants; Hold Defines Input Between Updates

- Sampling period T_s is the interval between records or updates. Zero-order hold (ZOH) keeps the last input value between updates; input does not vanish between samples.
- With constant input through an interval, integrate the continuous equation over that interval to obtain the next state from the current one.
- Sampling rate $f_s = 1/T_s$ differs from the solver's internal integration step. The solver may take several smaller steps within one input-hold period.



One discrete update under zero-order hold

Sampling period and input hold jointly define the discrete model's physical meaning.

4.4.5 Sampling, Hold, and Discretization: Exact Discretization of a First-Order Model

Solve the Differential Equation Over One Hold Interval

- **Hold:** from t_k to $t_k + T_s$, keep input u_k. Equation $\tau\dot{y} + y = Ku_k$ has equilibrium output Ku_k.

Within interval: error decays continuously

- **Derivation:** define error $\varepsilon = y - Ku_k$, which satisfies $\tau\dot{\varepsilon} + \varepsilon = 0$ in the interval and decays exponentially. Next state is target plus remaining error.

At interval end: target + remaining error

$$y(t_k + T_s) = Ku_k + (y_k - Ku_k)e^{-T_s/\tau}$$

- For $\tau = 0.05\,\text{s}$, $T_s = 0.01\,\text{s}$: $a = 0.81873075$, $1 - a = 0.18126925$.

Old-state weight 81.87% Target weight 18.13%

- **Meaning:** $a = e^{-T_s/\tau}$ is the fraction of old error retained; $1 - a$ is the fraction advanced toward this target. This integrates the interval, rather than approximating it with Euler.

4.4.5 Sampling, Hold, and Discretization: Iteration Versus the Continuous Solution

ZOH Iteration Matches the Continuous Solution at Sample Times

- **Example:** $K = 1, u_k = 1, y_0 = 0, T_s = 0.01\,\text{s}$. ZOH: $y_{k+1} = 0.8187y_k + 0.1813$; Euler: $y_{k+1} = 0.8y_k + 0.2$. Substitute the preceding result at every step.

k	Zero-order hold	Forward Euler	Continuous $1 - e^{-kT_s/\tau}$
1	0.1813	0.2000	0.1813
2	0.3297	0.3600	0.3297
3	0.4512	0.4880	0.4512

- **Check:** all three ZOH steps match the continuous values shown because the step input is held within each interval. Euler moves slightly farther per step and leads the continuous response initially.
- Old-state and steady-target weights sum to 1. With constant input, the discrete model still approaches Ku, matching the continuous steady state.

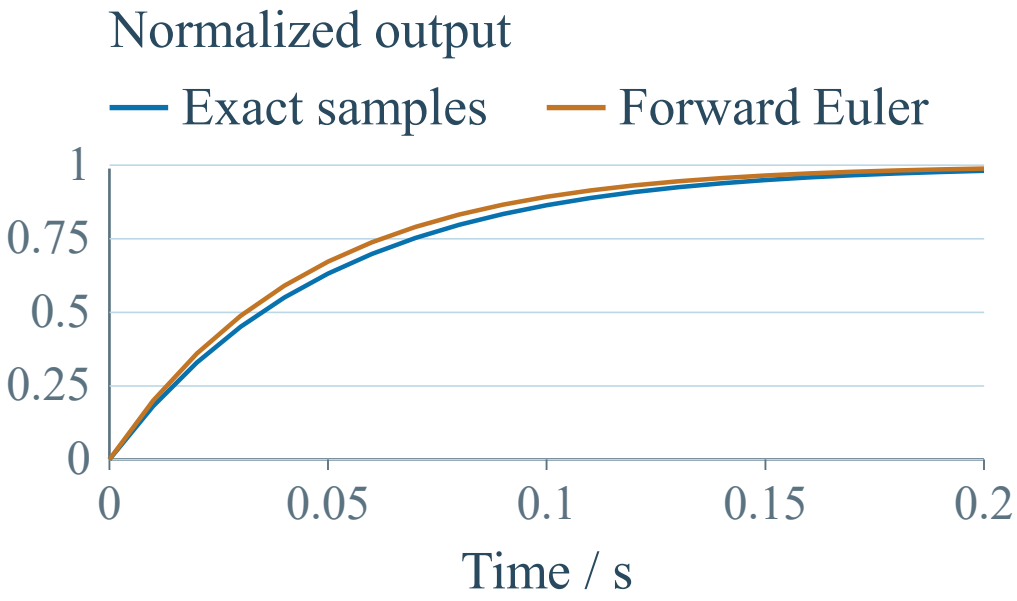
4.4.5 Sampling, Hold, and Discretization: Forward Euler Discretization and Stability

When Does Forward Euler Differ from Exact Discretization?

- Euler coefficient $a_E = 1 - T_s/\tau$. At $T_s = 0.01$ s, $a_E = 0.8$, versus exact 0.81873; the curves nearly overlap.
- Larger steps give larger differences: at $T_s = \tau$, $a_E = 0$, versus exact $e^{-1} \approx 0.368$. For $T_s > 2\tau$, $|a_E| > 1$, so Euler diverges.

$$y_{k+1} = (1 - T_s/\tau)y_k + (T_s/\tau)Ku_k$$

$$|1 - T_s/\tau| < 1 \Leftrightarrow 0 < T_s < 2\tau$$



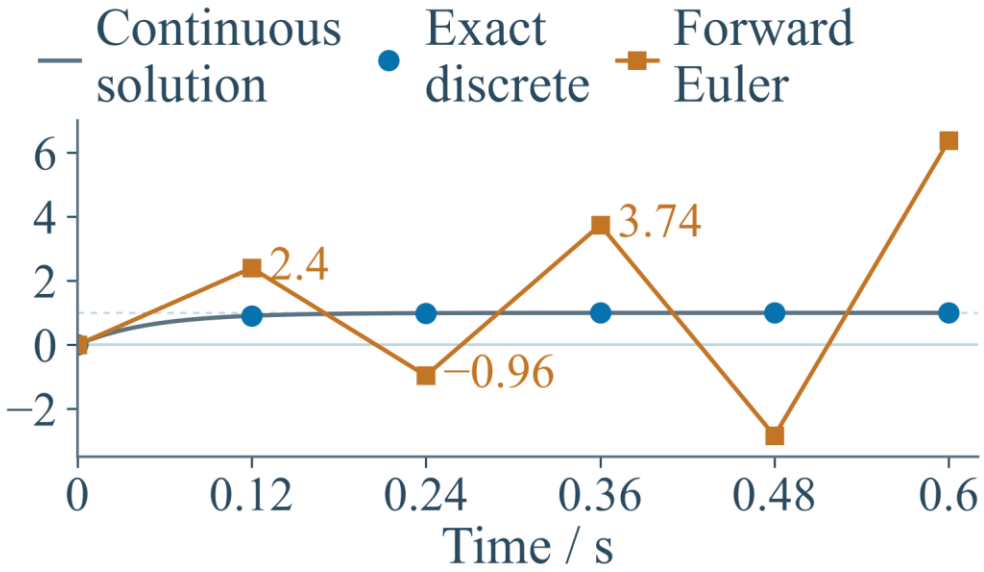
Two updates under the same step input

Discrete approximation error depends on sampling interval relative to the dynamic time scale.

4.4.5 Sampling, Hold, and Discretization: Crossing the Stability Limit

Above 2τ, Euler Diverges Alternately; Exact Updates Converge

- **Example:** $\tau = 0.05\,\text{s}$, $T_s = 0.12\,\text{s}$, $u = 1$, $y_0 = 0$. Euler: $y_{k+1} = -1.4y_k + 2.4$, giving 2.4, −0.96, 3.74, −2.84, …
- **Exact:** $y_{k+1} = 0.0907y_k + 0.9093$, giving 0.909, 0.992, 0.999, approaching 1 monotonically.
- **Conclusion:** error alternates and grows by factor −1.4 each step. The continuous plant is stable; the method and step size cause divergence.



Unit step at $T_s = 0.12\,\text{s}$ (teaching example, not platform measurements)

A stable continuous model does not guarantee stable numerical updates at every step size.

4.4.5 Sampling, Hold, and Discretization: Exact State-Space Discretization

From General Continuous State Space to Discrete Time

- A constant-coefficient linear model under ZOH satisfies $\mathbf{x}_{k+1} = A_d\mathbf{x}_k + B_d\mathbf{u}_k$. The exponential advances internal dynamics one sampling period; the interval integral accumulates input effects.

$$A_d = e^{AT_s}, \qquad B_d = \int_0^{T_s} e^{A\eta}\, B\, \mathrm{d}\eta$$

- If A is invertible, use $B_d = A^{-1}(A_d - I)B$. With position or velocity integrators, A is singular. The vertical model's first column of A is zero; for $T_s = 0.01$ s, integration or c2d gives:

$$A_d \approx \begin{bmatrix} 1 & -3.59 \times 10^{-4} \\ 0 & 0.8187 \end{bmatrix}, \quad B_d \approx \begin{bmatrix} -0.0371 \\ 181.27 \end{bmatrix}$$

Matrix exponential and integration give the ZOH relation without requiring an invertible system matrix.

4.4.5 Sampling, Hold, and Discretization: Reproduce the First-Order Example with c2d

Check ZOH Coefficients with c2d

- Given: $0.05\dot{y} + y = u$, zero initial value, sampling period 0.01 s, input constant within each interval. This is a delay-free first-order linear model.
- Run in MATLAB (requires Control System Toolbox):

```
sysc = tf(1,[0.05 1]);   Ts = 0.01;   sysd = c2d(sysc,Ts,'zoh')
```

- Expected: $0.1813/(z - 0.8187)$ and Sample time: 0.01 seconds. Symbol z shifts forward one period.
- Analytical check: $a = e^{-0.01/0.05} = 0.81873075$, $b = 1 - a = 0.18126925$. From $(z - a)Y = bU$, obtain $y_{k+1} = ay_k + bu_k$.

c2d documentation: https://www.mathworks.com/help/control/ref/dynamicsystem.c2d.html

Match tool output to analytical coefficients; record both method and sampling period.

Class Exercise

New Data: Recalculate Slope, Discrete Stability, and Time Constant

- 1. Use fitted-example $c_T \approx 2.657 \times 10^{-5}$ from slide section 4.6 and hover speed $\omega_0 \approx 429.4\,\mathrm{rad/s}$. Find total-thrust slope $8c_T\omega_0$, compare with 0.0792, and state how the vertical state matrix and cascade transfer-function gain change.
- 2. For $\tau = 0.05\,\mathrm{s}$ and T_s of 0.075 s and 0.12 s, calculate Euler coefficient a_E and exact $a = e^{-T_s/\tau}$. Classify monotonic convergence, alternating convergence, or divergence.
- 3. A motor step test measures 300, 615, 806, 921, 992 rad/s at $t = 0$, 0.04, 0.08, 0.12, 0.16 s; final speed is about 1100 rad/s. Estimate τ.
- **Extension:** textbook Exercise 8 (hover linearization and discretization workflow) is discussed in slide section 4.12, Exercise 8.

Reference Answers

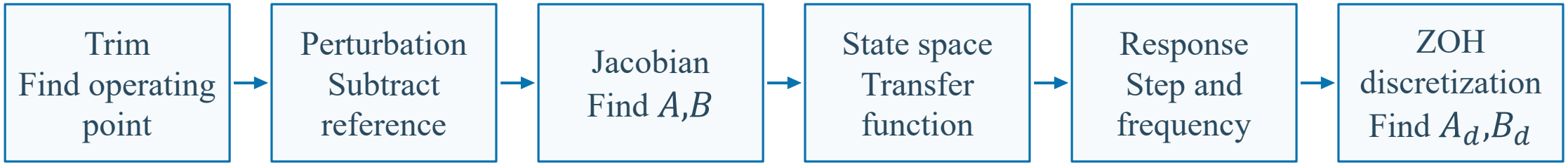
Exercise Results and the Reasoning Behind Them

- 1. $8c_T\omega_0 \approx 0.0913\,\mathrm{N/(rad/s)}$, about 1.15×0.0792. State-matrix row 1 changes from -0.039598 to about -0.0456 (0.0913 divided by m). $B = [0,\ 20000]^T$ contains only motor parameters and stays unchanged. Cascade gain changes from -39.598 to about -45.6.

- 2. $T_s = 0.075\,\mathrm{s}$: $a_E = -0.5$, alternating convergence; $a = e^{-1.5} \approx 0.223$, monotonic convergence. $T_s = 0.12\,\mathrm{s}$: $a_E = -1.4$, alternating divergence; $a = e^{-2.4} \approx 0.091$, monotonic convergence.
- **3. Change:** 1100−300=800 rad/s; 63.2% is 506 rad/s. Add the start to obtain about 806 rad/s, giving $\tau \approx 0.08\,\mathrm{s}$. Taking 63.2% of 1100 directly (695 rad/s) incorrectly gives about 0.055 s.

Section Summary

A Method Chain from Trim to a Discrete Model



- **Increments and absolute values:** linear models compute deviations from the operating point. Add the reference back before sending values to an interface.
- **Units:** angles in radians, 1°=0.01745 rad; angular frequency in rad/s, $\Omega = 2\pi f$.
- **Conditions:** transfer functions assume zero initial state; forward Euler needs $0 < T_s < 2\tau$. If A is singular, find B_d by integration or c2d.

Chapter 5 uses these linear and discrete models for state feedback and Kalman filtering.