Chapter Outline

Rigid-Body Dynamics and Motion Solvers

Chapter 4
Vehicle Motion Modeling and Simulation
4.1 Modeling Fundamentals and Kinematics
4.2 Rigid-Body Dynamics and Solvers
4.3 Propulsion, Forces, and Observations
4.4 Trim, Linearization, and Dynamics
4.5 Multicopter Modeling and Simulation
4.6 Measurement and Static Identification
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4.8 Typical Vehicles and Extensions
4.9 Modeling Tools and Deployment
4.10 API and SDK Modeling Interfaces
4.11 Examples and Integrated Practice
4.12 Exercises and Course Projects

5 Units in This Section

4.2.1 Hovering and Vertical Motion

Textbook: Sec. 4.1.4.1

4.2.2 Differential Thrust and Rotation

Textbook: Sec. 4.1.4.1

4.2.3 Tilt and Three-Dimensional Dynamics

Textbook: Secs. 4.1.3.2 and 4.1.4.1

4.2.4 Continuous Equations and Numerical Integration

Forward Euler, step-size error, and numerical stability

4.2.5 Analytical Checks and Exercises

Check balance, single-axis rotation, and error sources

Section Introduction

Sum Forces and Torques, Then Compute Motion

- **Inputs:** Known rotor thrusts and torques about mass center. Values: $m = 2\,\text{kg}$, $g = 9.8\ \text{m/s}^2$, 0.25 m moment arm, $J_y = 0.05\,\text{kg}\cdot\text{m}^2$, 19.6 N hover thrust.
- **Dynamics outputs:** Newton's equation gives linear acceleration; Euler's dynamics gives angular acceleration. Keep inertia and coupling terms distinct.
- **Reconnect kinematics:** Velocity updates position; body angular velocity updates the quaternion (Section 4.1). Section 4.5 implements and simulates these equations.

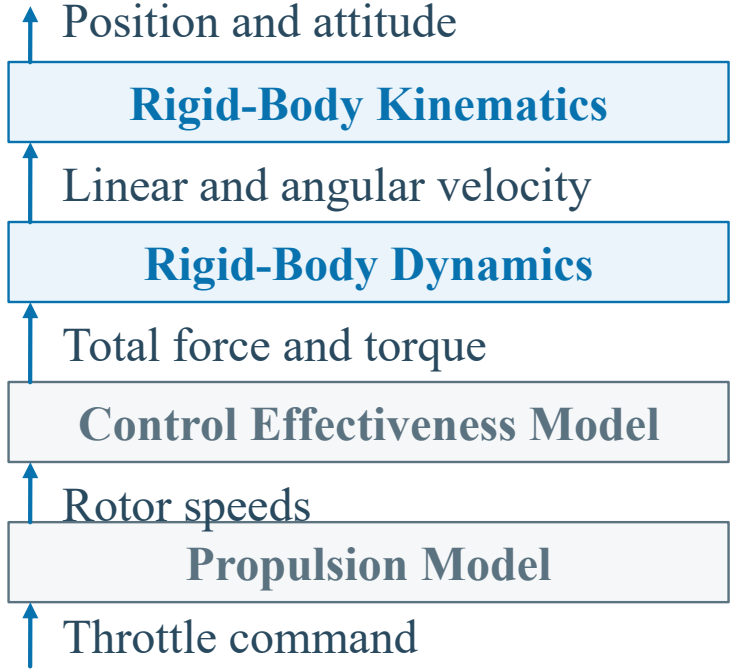


Figure: Four layers of the motion model

Dynamics converts external forces and torques into state derivatives; kinematics accumulates them into a trajectory.

4.2.1 Hovering and Vertical Motion: Force and Acceleration

Net Vertical Force Changes Velocity

- **Hover:** Four rotors each supply 4.9 N, so $T = mg = 19.6$ N. Holding position also requires level attitude, zero net torque, and zero linear and angular velocity.
- **Rates:** Acceleration is velocity change per second; the position derivative is position change per second. A dot denotes a time derivative. Integration accumulates rates into states.
- **Increase thrust:** Four rotors at 5.4 N each give 2 N net upward force. Dividing by 2 kg gives $a_D = -1\,\mathrm{m/s^2}$. In the north-east-down (NED) frame, D points down, so upward is negative.

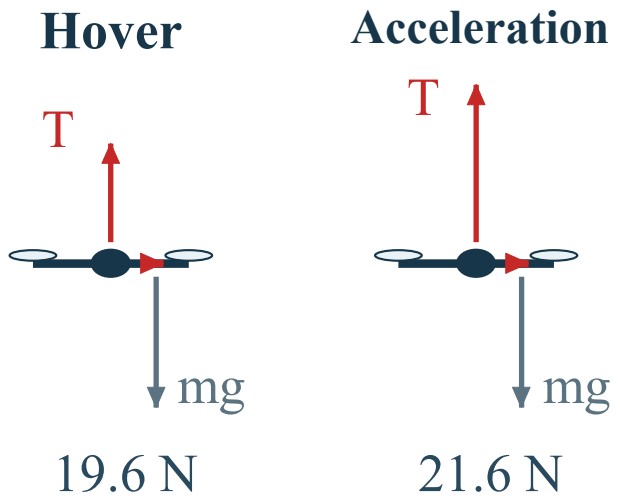


Figure: Thrust up; weight down

$$T = T_1 + T_2 + T_3 + T_4, \qquad a_D = \dot{v}_D = g - \frac{T}{m}, \qquad \dot{z}_D = v_D$$

Thrust first changes acceleration; accumulated velocity changes then change height.

4.2.1 Hovering and Vertical Motion: Analytical Results and Curves

How Far Does Constant Thrust Lift the Vehicle in 1 s?

- **Start the clock at the beginning of acceleration:** $z_D(0) = -1\,\text{m}$ and $v_D(0) = 0$. Total thrust is 21.6 N, so $a_D = -1\,\text{m/s}^2$.
- **After 1 s:** $v_D = -1\,\text{m/s}$ and $z_D = -1.5\,\text{m}$, meaning a height of 1.5 m and an upward speed of 1 m/s.

- **Accumulate displacement:** The triangular area is a 0.5 m rise. The analytical formula assumes constant acceleration; numerical results should approach it as the step shrinks.

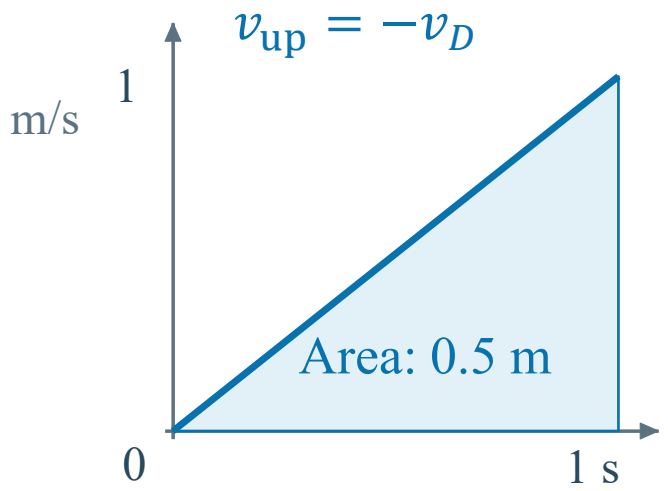


Figure: Area is displacement

$$v_D(t) = v_D(0) + a_D t, \qquad z_D(t) = z_D(0) + v_D(0)t + \frac{1}{2}a_D t^2$$

Given the initial state and acceleration, calculate velocity and position at any time.

4.2.1 Hovering and Vertical Motion: Motion and Curves

To Rise and Stop, Accelerate Then Brake

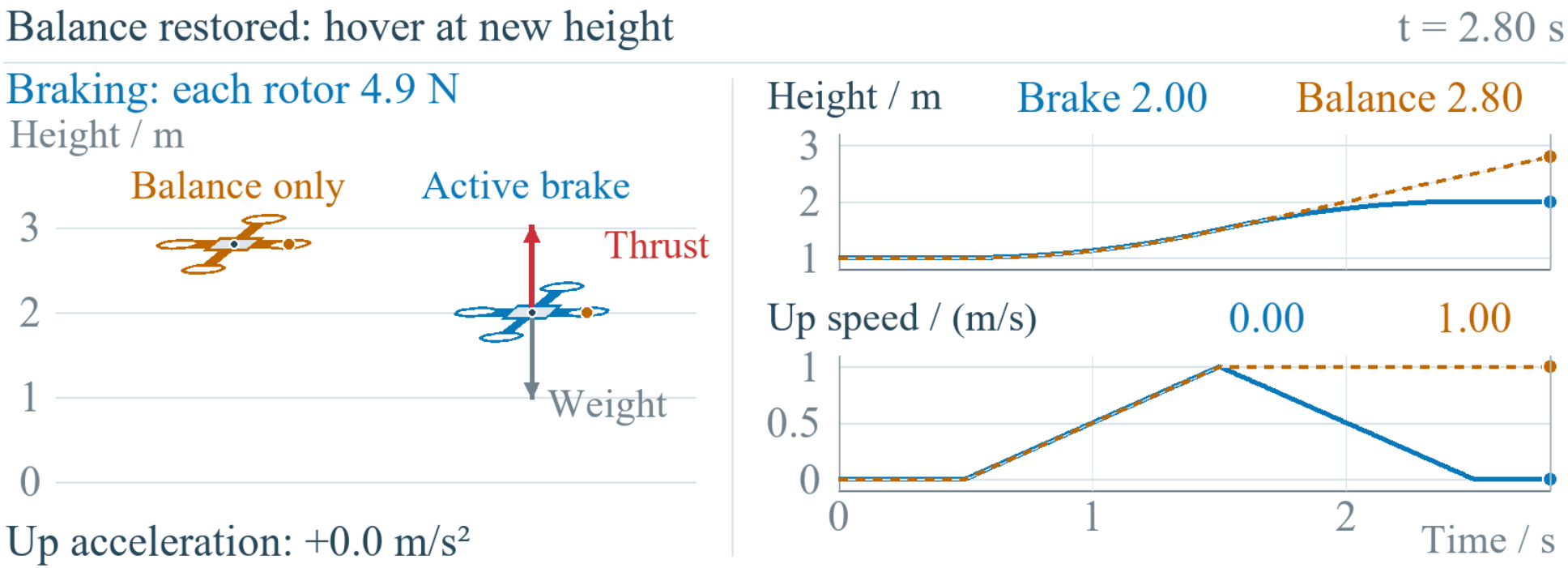


Figure: Zero drag. Accelerate at 21.6 N and brake at 17.6 N; restoring 19.6 N early leaves the vehicle climbing.

Hover thrust removes acceleration; stopping the climb first requires removing upward velocity.

4.2.1 Hovering and Vertical Motion: States and Control Targets

Stopping Requires Both Target Height and Zero Velocity

- **After acceleration:** Height is 1.5 m and upward speed is 1 m/s. With zero drag, restoring $T = 19.6$ N only balances force; the vehicle keeps climbing.
- **Braking:** Restart the clock. Four rotors each supply 4.4 N, totaling 17.6 N, with $a_D = +1\ \mathrm{m/s^2}$. Net force opposes velocity, so the vehicle slows.
- Brake for 1 s: rise 0.5 m more. At zero velocity, restore 19.6 N to hold 2 m height. Mass changes or disturbances need state-feedback correction.

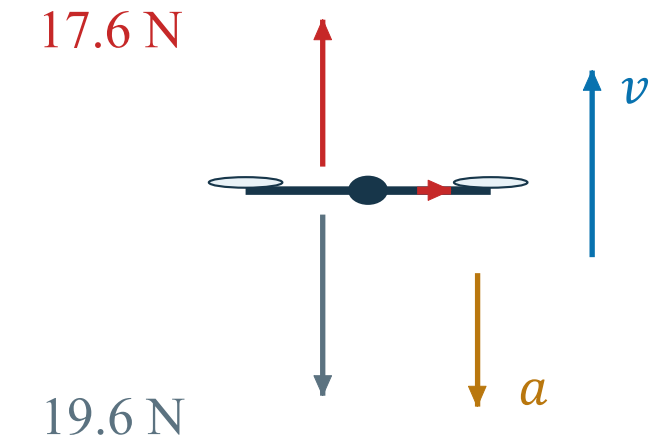


Figure: Rising; downward accel.

$$v_{D,\mathrm{end}} = -1 + 1 \times 1 = 0\ \mathrm{m/s}, \qquad \Delta z_D = -1 \times 1 + \frac{1}{2} \times 1 \times 1^2 = -0.5\ \mathrm{m}$$

The motion model predicts responses; the controller selects inputs from the target and current state.

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4.2.2 Differential Thrust and Rotation: Mass and Moments about the Center of Mass

Mass, Center of Mass, and Force Location Set the Response

- **Mass and center of mass:** m measures translational inertia. The center of mass is the mass-weighted position. Place the body origin there; the geometric center and autopilot mounting point may differ.
- **Torque:** ρ runs from mass center to force point. Sum $\rho \times F$ and direct couples τ_i (e.g., rotor reaction torque). A force through mass center translates the body with zero moment-arm torque.

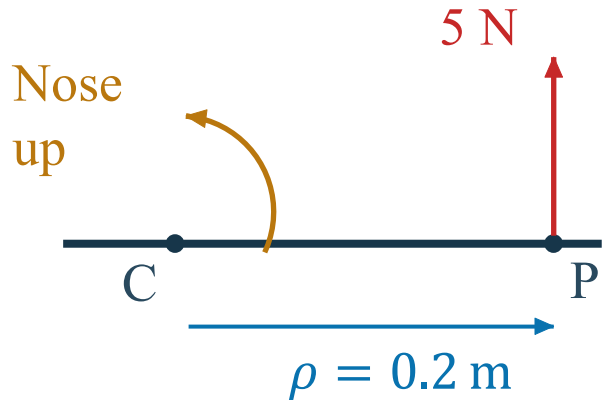


Figure: Center of mass and force application point

$$\mathbf{r}_C = \frac{1}{m}\int \mathbf{r}\,\mathrm{d}m, \qquad \mathbf{M}_b = \sum_i \boldsymbol{\rho}_{i,b} \times \mathbf{F}_{i,b} + \sum_i \boldsymbol{\tau}_{i,b}$$

- **Example:** A 5 N upward force 0.2 m forward gives positive (nose-up) pitch torque.

$$[0.2,0,0]^{\mathrm{T}} \times [0,0,-5]^{\mathrm{T}} = [0,1,0]^{\mathrm{T}}\ \mathrm{N\,m}$$

External force drives translation; its lever arm about the center of mass sets rotation.

4.2.2 Differential Thrust and Rotation: A Single-Axis Calculation

How Does Differential Thrust Produce Pitch?

- Differential input: Each side rotor stays at 4.9 N. Total thrust remains 19.6 N; unequal front and rear moments give negative (nose-down) pitch torque.
- **Single right-axis rotation:** $J_y = 0.05\ \mathrm{kg\,m^2}$. Inertia is mass's rotational counterpart. Here pitch rate equals angular velocity about that axis; angular acceleration is torque divided by inertia.

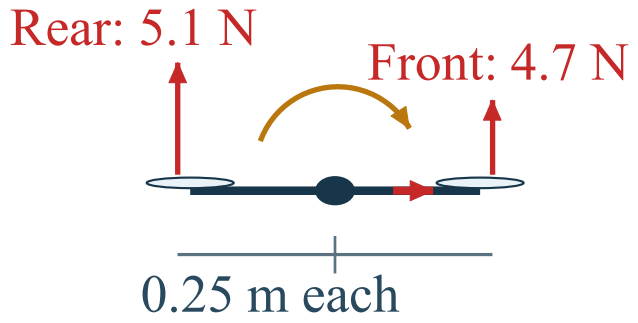


Figure: Weaker front thrust pitches the nose down

① Calculate torque

$M_y = l(T_{\text{front}} - T_{\text{rear}})$
$= 0.25 \times (4.7 - 5.1)$
$= -0.1\ \mathrm{N \cdot m}$

→

② Angular acceleration

$\dot{\omega}_y = M_y / J_y$
$= -0.1/0.05 = -2\ \mathrm{rad/s^2}$
Single-axis: $\dot{\theta} = \omega_y$

→

③ Accelerate, then brake

− then + torque: 0.3 s each.
ω_y returns to 0;
$\theta = -0.18$ rad (about −10.3°).

Torque first changes angular velocity; integrating angular velocity then changes attitude.

4.2.2 Differential Thrust and Rotation: Torque and Attitude

Removing Torque Does Not Stop Rotation

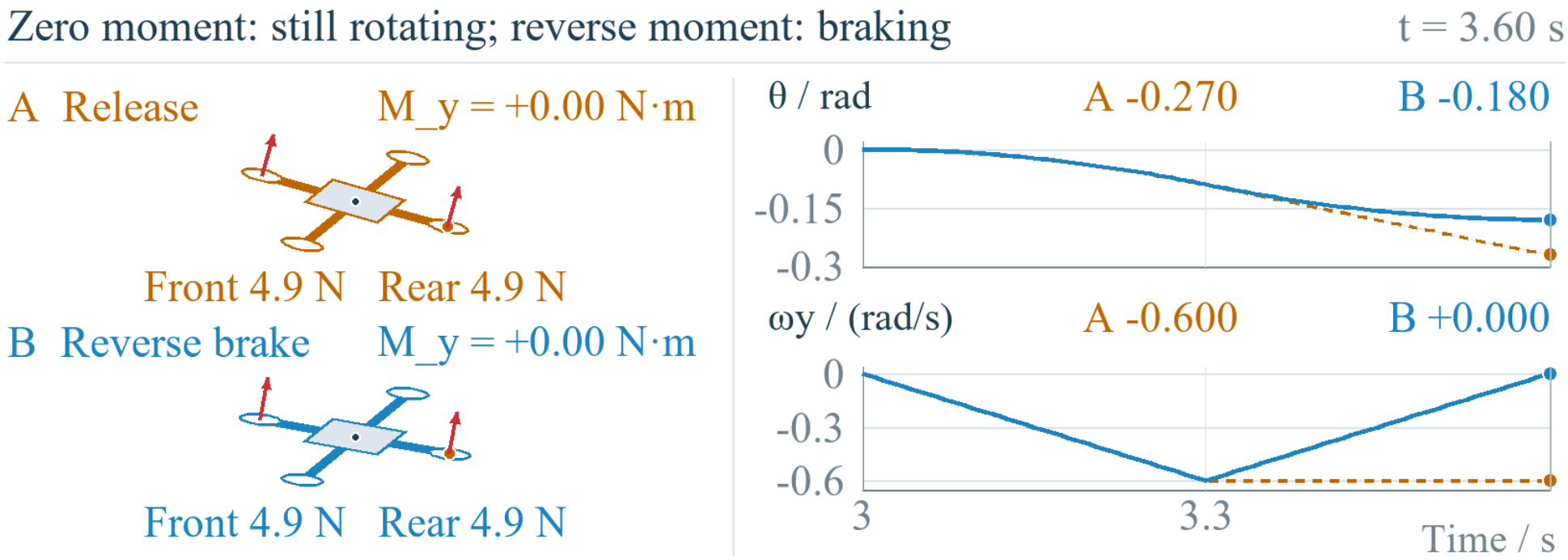


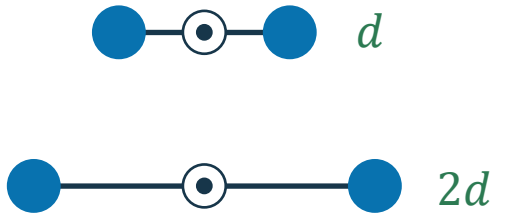
Figure: Same initial state: removing torque leaves rotation; opposite torque brings angular velocity to zero.

To hold a tilted attitude, first reduce angular velocity to zero.

4.2.2 Differential Thrust and Rotation: Inertia, Principal Axes, and Positive Definiteness

Inertia Describes Mass Distribution and Rotation

- **Meaning:** J depends on mass distribution about mass center; units $\mathrm{kg} \cdot \mathrm{m}^2$. More distance perpendicular to x increases J_x. Double a point mass's distance: quadruple inertia. The same torque gives less angular acceleration.



Same masses, farther out

$$J = \int [(\boldsymbol{\rho}^{\mathsf{T}}\boldsymbol{\rho})I - \boldsymbol{\rho}\boldsymbol{\rho}^{\mathsf{T}}]\,\mathrm{d}m, \quad J_{xx} = \int (\rho_y^2 + \rho_z^2)\,\mathrm{d}m$$

- **Symmetry and principal axes:** J is symmetric, with six independent entries. Off-diagonals include $-\int \rho_x \rho_y \, \mathrm{d}m$. Principal axes diagonalize it; body axes need not coincide with them.
- **Constraints:** Angular momentum $\mathbf{H}_b$ and rotational energy E_{rot} below. Positive-definite J: positive energy for nonzero rotation. Each principal moment ≤ sum of the other two.

$$\mathbf{H}_b = J\boldsymbol{\omega}_b, \qquad E_{\mathrm{rot}} = 1/2\,\boldsymbol{\omega}_b^{\mathsf{T}} J \boldsymbol{\omega}_b$$

Inertia describes mass distribution and determines angular momentum, rotational energy, and angular acceleration.

4.2.2 Differential Thrust and Rotation: Angular Momentum Balance

Euler's Equation Includes Angular Momentum Coupling

- **Assumptions:** Body origin at the center of mass; fixed mass distribution and constant J. $\boldsymbol{\omega}_b = [p, q, r]^\mathsf{T}$ gives angular velocities about forward, right, and down. $\mathbf{M}_b$ is body-frame torque about the same center of mass.
- **Coupling:** In inertial Earth axes, external torque changes angular momentum. In rotating body axes, both components and axes change.

$$\mathbf{M}_b = J\dot{\boldsymbol{\omega}}_b + \boldsymbol{\omega}_b \times (J\boldsymbol{\omega}_b)$$

Two angular-momentum changes $J\dot{\boldsymbol{\omega}}_b$: Component change $\boldsymbol{\omega}_b \times (J\boldsymbol{\omega}_b)$: Rotating coordinates

- Diagonal J: Roll still has pitch-rate × yaw-rate coupling. It vanishes only if a relevant rate is zero or corresponding principal moments are equal. Model internal-rotor momentum separately.

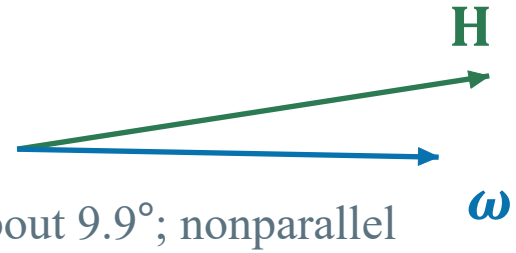
$$J_x\dot{p} = M_x + (J_y - J_z)qr$$

The rotational equation accounts for external torque and angular momentum coupling in rotating coordinates.

4.2.2 Differential Thrust and Rotation: An Inertial-Coupling Example

Zero External Torque Can Still Change Body Rates

- **Instantaneous conditions:** $J = \mathrm{diag}(0.02, 0.03, 0.04)$ kg·m², $\boldsymbol{\omega} = [1,2,3]^{\mathsf{T}}$ rad/s, and zero external torque. Unequal principal moments make angular momentum $\mathbf{H} = J\boldsymbol{\omega}$ nonparallel to angular velocity.



- **Calculate in order:** $\mathbf{H} = [0.02, 0.06, 0.12]^{\mathsf{T}}$; $\boldsymbol{\omega} \times \mathbf{H} = [0.06, -0.06, 0.02]^{\mathsf{T}}$ N·m. Negate each component and divide by its moment of inertia:

$$\dot{\boldsymbol{\omega}}_b = -J^{-1}(\boldsymbol{\omega}_b \times J\boldsymbol{\omega}_b) = [-3, 2, -0.5]^{\mathsf{T}}\ \mathrm{rad/s^2}$$

- **Conservation checks:** Inertial-space angular momentum stays constant, although body-axis components may change. Rotational kinetic energy is 0.25 J, with $\dot{E}_{\mathrm{rot}} = \boldsymbol{\omega}^{\mathsf{T}}\mathbf{M} = 0$.

$$E_{\mathrm{rot}} = 1/2\,(0.02 \times 1^2 + 0.03 \times 2^2 + 0.04 \times 3^2) = 0.25\,\mathrm{J}$$

With zero external torque, inertial-space angular momentum is conserved; individual body-rate components need not be.

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4.2.3 Tilt and Three-Dimensional Dynamics: Observed Motion

Tilt Causes Translation; Leveling Leaves Drift

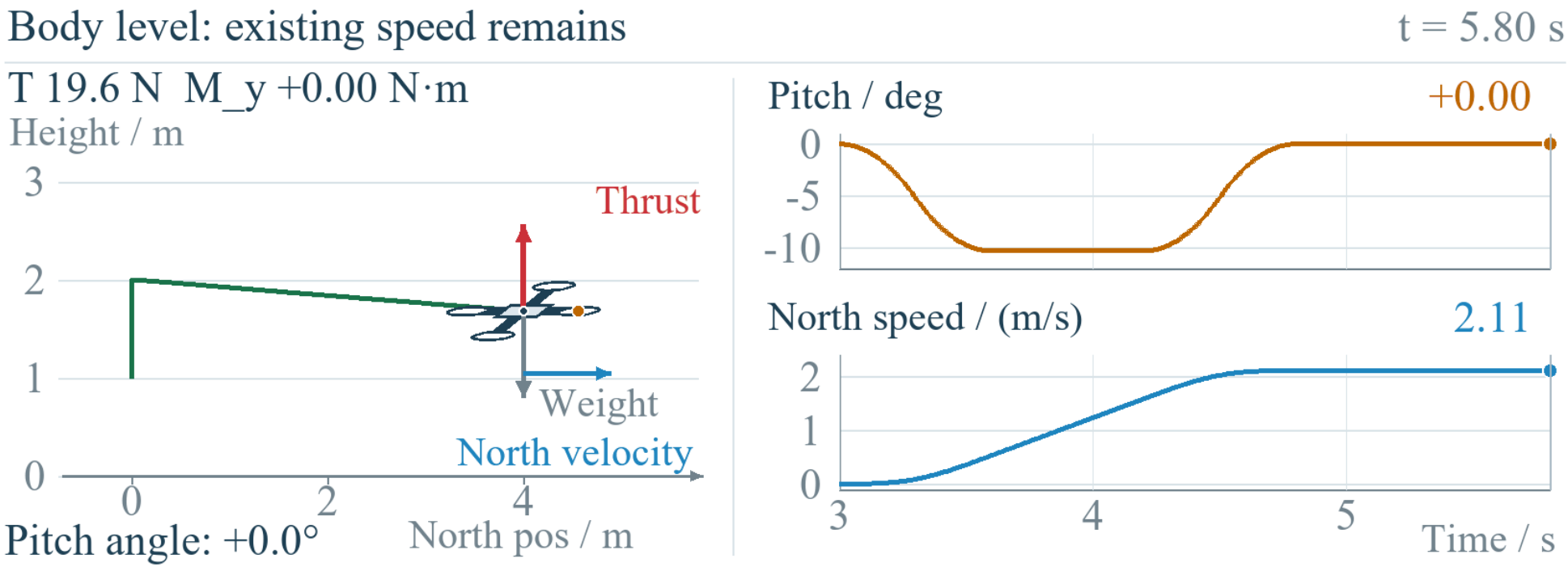


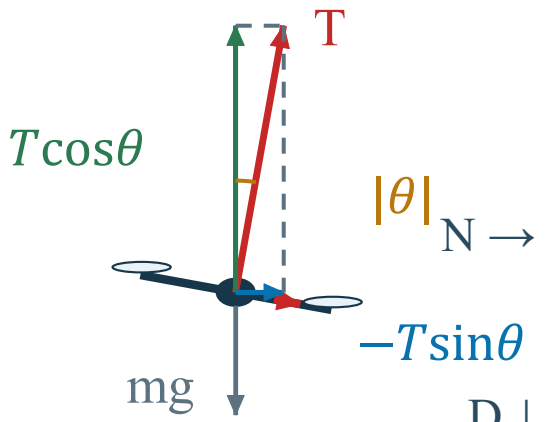
Figure: With zero drag, leveling leaves zero net horizontal force, so the vehicle drifts north at constant speed.

Attitude sets thrust direction; translational velocity also depends on the earlier force history.

4.2.3 Tilt and Three-Dimensional Dynamics: Transforming Thrust

Resolve Body Thrust into Earth-Frame Components

- **Notation:** b/e = body/Earth; N/D = north/down. $F_{\mathrm{ng},b}$ is net nongravitational force—only thrust here. R maps body to Earth directions.
- **Pitch plane:** Zero roll and yaw. Combine both thrust components with downward weight. Changing coordinates changes representation only; it adds no thrust.
- At 19.6 N and −0.18 rad: about $1.75\,\mathrm{m/s^2}$ northward and $0.158\,\mathrm{m/s^2}$ downward. Vertical thrust is insufficient.



One thrust, two components

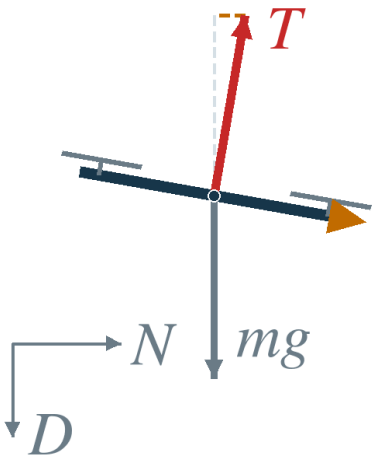
$$\mathbf{F}_{ng,b} = [0,0,-T]^{\mathsf{T}}, \qquad \mathbf{F}_{ng,e} = R\mathbf{F}_{ng,b}$$

$$a_N = -\frac{T}{m}\sin\theta, \qquad a_D = g - \frac{T}{m}\cos\theta$$

The same thrust changes direction with attitude, affecting both horizontal motion and height.

4.2.3 Tilt and Three-Dimensional Dynamics: An Instantaneous Force Sweep

At 19.6 N, Nose-Down Tilt Raises Both Accelerations



$T = 19.6$ N
$\theta = -10.0^\circ$
nose down
$a_N = +1.702$ m/s^2
$a_D = +0.149$ m/s^2
acceleration
north
Downward accel.

Animation: Instantaneous acceleration from 0 to −10° pitch (teaching example)

- **Sweep:** $m = 2$ kg, $T = 19.6$ N. Vary pitch θ from 0 to −10° (nose down); compute instantaneous acceleration at each angle.

$$a_N = -\frac{T}{m}\sin\theta\,, \quad a_D = g - \frac{T}{m}\cos\theta$$

- **Read the plot:** At $\theta = -10^\circ$, $a_N \approx +1.70$ m/s^2, northward; $a_D \approx +0.149$ m/s^2, downward. Greater tilt increases northward acceleration and the vertical thrust deficit.

This instantaneous force sweep shows changing directions; it does not integrate a flight trajectory.

4.2.3 Tilt and Three-Dimensional Dynamics: Conditions for Holding Height

Holding Height Also Requires Zero Vertical Velocity

- **Tilted height holding:** Retain −0.18 rad tilt. Matching vertical thrust to weight removes vertical acceleration only; any existing descent velocity remains.
- **Braking condition:** First make upward force exceed weight to reduce descent velocity to zero, then restore balanced thrust to hold height.
- **Horizontal coupling:** More thrust also increases horizontal acceleration. Choose inputs using velocity and attitude as well as the position target.

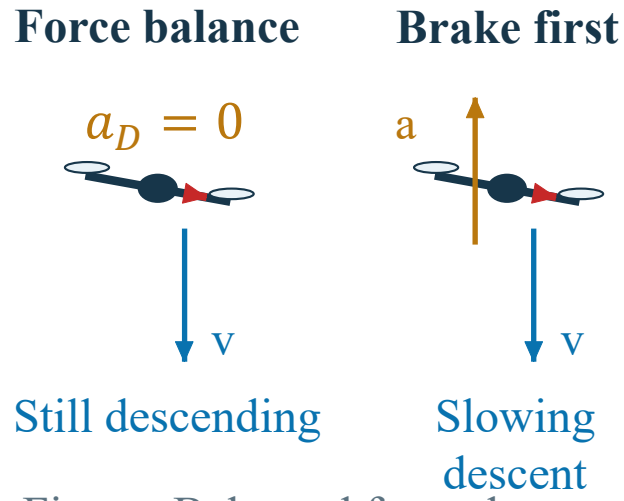


Figure: Balanced force does not imply zero velocity

$$T\cos\theta = mg \quad \Rightarrow \quad T = \frac{mg}{\cos\theta} \approx 19.92\ \text{N} \quad (\theta = -0.18\ \text{rad})$$

Constant height requires both zero net vertical force and zero vertical velocity.

4.2.3 Tilt and Three-Dimensional Dynamics: Rigid-Body Subsystem Signals

Connect Actuators, Environment, Forces, and Body Dynamics

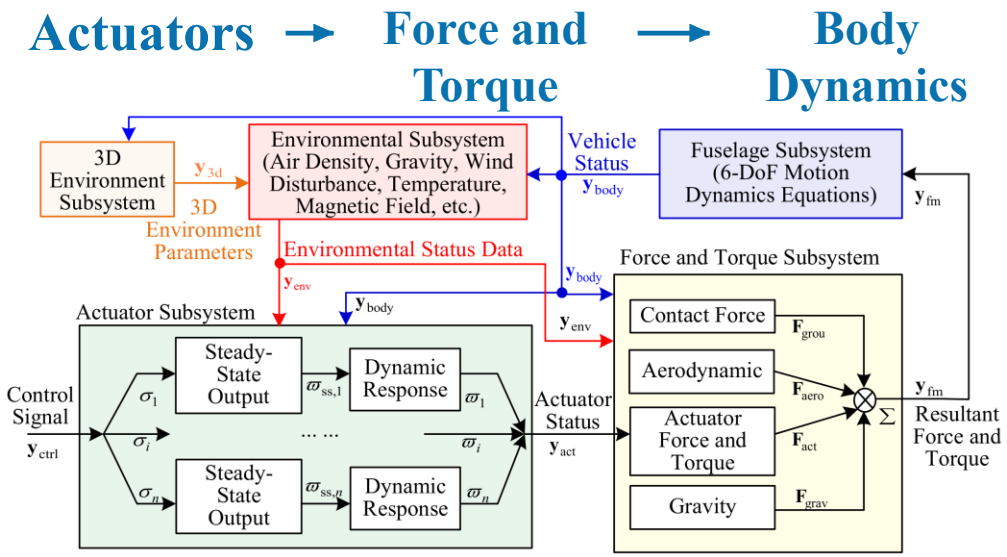


Figure 4.8 Rigid-Body Subsystem Interconnections

- **Actuators:** Commands → actual rotor speeds. Environment: Body states and 3D scene → wind, gravity, and other conditions.
- **Force/torque synthesis:** Sum propulsion, aerodynamic, contact, and gravity contributions using actuator, environment, and body-state data.
- **Body dynamics and feedback:** Net forces and torques update motion states, which feed actuator, environment, and force calculations. The total force shown already includes gravity.

Actuators and environment produce loads; body states feed back into the next force calculation.

4.2.3 Tilt and Three-Dimensional Dynamics: Figure 4.8 Detail: Actuators and Forces

Actuators Produce Rotor Speed; Four Loads Meet at the Sum

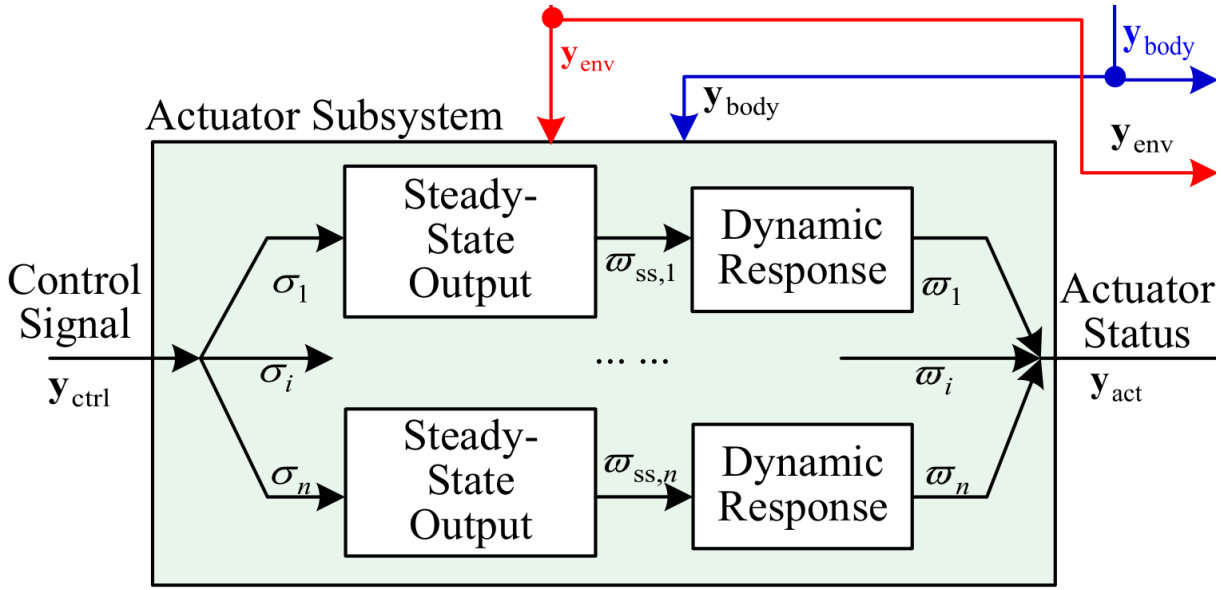


Figure 4.8 (detail) Actuator Subsystem

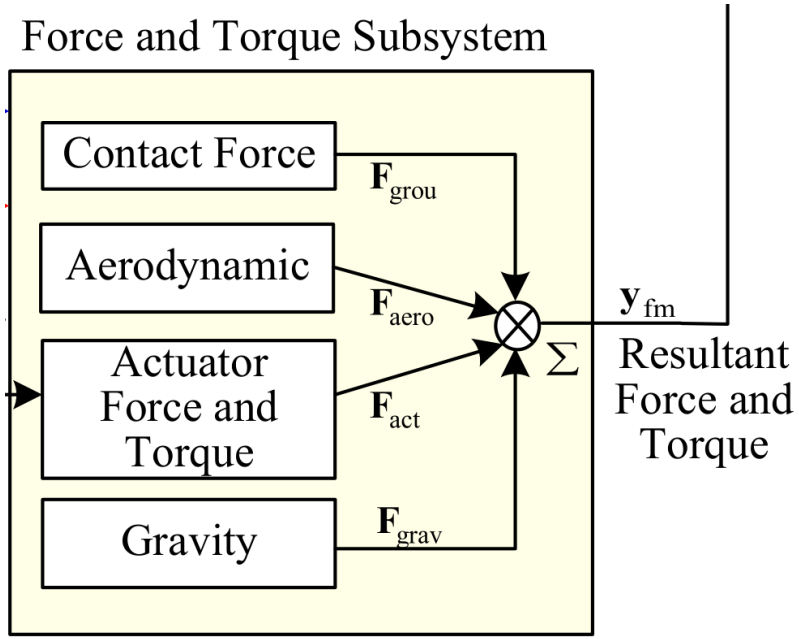


Figure 4.8 (detail) Force and Torque Synthesis Subsystem

- **Read:** σ_i passes through steady-state output and dynamic response to give speed ϖ_i. Four force contributions sum at Σ to give $\mathbf{y}_{\mathrm{fm}}$.

4.2.3 Tilt and Three-Dimensional Dynamics: Figure 4.8 Detail: Environment and Feedback

Body States Feed Back to Three Subsystems

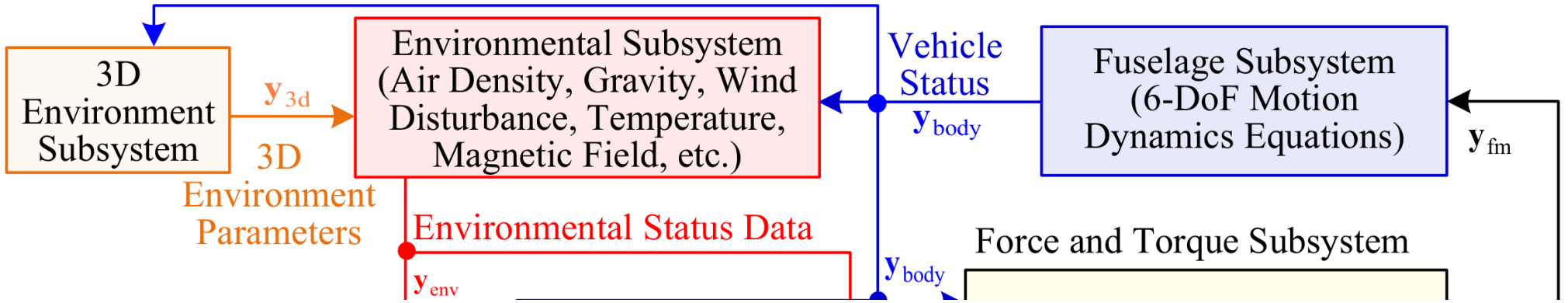


Figure 4.8 (detail) Environment, 3D Environment, and Body-State Feedback

- **3D environment:** Parameters $\mathbf{y}_{3d}$ go only to the environment subsystem.
- **Environment subsystem:** Use $\mathbf{y}_{3d}$ and vehicle states $\mathbf{y}_{body}$ to calculate wind, gravity, and other environmental states $\mathbf{y}_{env}$ for actuator and force calculations.
- **Body dynamics subsystem:** Solve the six-degree-of-freedom motion equations using net force and torque $\mathbf{y}_{fm}$. Return updated $\mathbf{y}_{body}$ to the other three subsystems.

4.2.3 Tilt and Three-Dimensional Dynamics: Two Equivalent Gravity Conventions

Include Gravity in Force or Add It as Acceleration

- **Common input:** Net nongravitational force includes propulsion, aerodynamics (including drag), and contact. Textbook Figure 4.8 uses convention B: total force. Only the introductory baseline sets drag to zero.

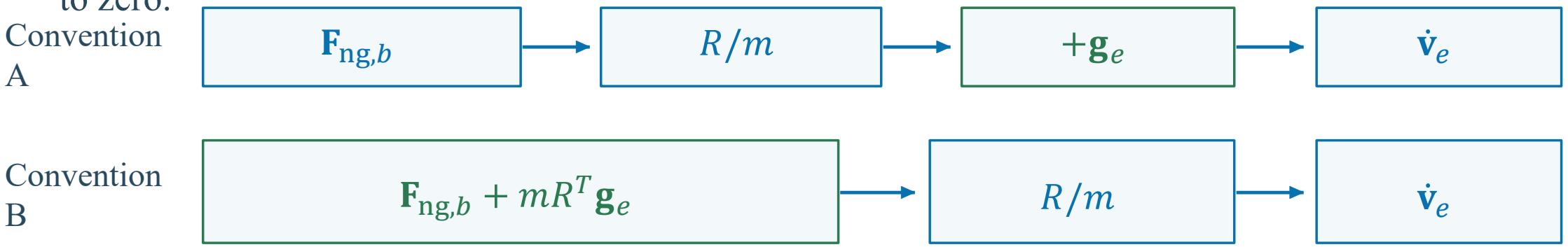


Figure: One gravity contribution, two entry points, equal results

- **Free-fall check:** Remove all nongravitational forces; both routes give $a_D = 9.8\,\mathrm{m/s^2}$. A result of 19.6 usually indicates double-counted gravity; 0 may indicate omitted gravity.

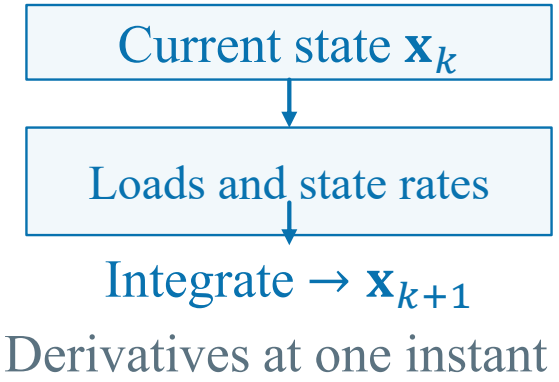
Both routes give the same acceleration. Count gravity once in each route.

4.2.3 Tilt and Three-Dimensional Dynamics: A Six-Degree-of-Freedom State Model

Combine Kinematics and Dynamics into State Equations

- **Assumptions:** Inertial local NED; body origin at mass center; constant mass m and inertia J.

$$\dot{\mathbf{r}}_e = \mathbf{v}_e$$
$$\dot{\mathbf{v}}_e = \mathbf{g}_e + R(\mathbf{q})\mathbf{F}_{\mathrm{ng,b}}/m$$
$$\dot{\mathbf{q}} = 1/2\,\mathbf{q} \otimes [0, \boldsymbol{\omega}_b^{\mathsf{T}}]^{\mathsf{T}}$$
$$\dot{\boldsymbol{\omega}}_b = J^{-1}[\mathbf{M}_b - \boldsymbol{\omega}_b \times (J\boldsymbol{\omega}_b)]$$



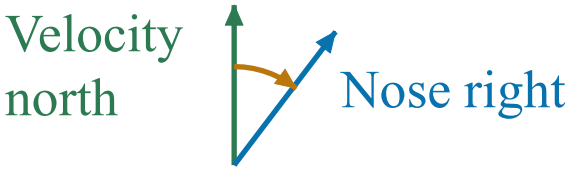
- **Inputs:** $\mathbf{F}_{\mathrm{ng},b}$ sums nongravitational external forces, including drag; $\mathbf{M}_b$ sums external torques about the center of mass, including rotational damping. $R(\mathbf{q})$ maps body to Earth components. Four equation groups advance position, velocity, attitude, and angular velocity.
- **States:** Store 13 rigid-body values (12 independent). Differentiate any actuator or sensor dynamics at the same instant and advance all states together.

Kinematics links states; dynamics uses actual subsystem forces/torques for accelerations.

4.2.3 Tilt and Three-Dimensional Dynamics: Derivatives in Rotating Coordinates

Rotating Axes Add a Body-Velocity Derivative Term

- **Product rule:** Both the components and coordinate axes in $\mathbf{v}_e = R\mathbf{v}_b$ change. Substitute $\dot{R} = R[\boldsymbol{\omega}_b]_\times$ to obtain:



$$\dot{\mathbf{v}}_e = \dot{R}\mathbf{v}_b + R\dot{\mathbf{v}}_b = R(\boldsymbol{\omega}_b \times \mathbf{v}_b + \dot{\mathbf{v}}_b)$$

$$\dot{\mathbf{v}}_b = \mathbf{F}_{\mathrm{ng},b}/m + R^{\mathsf{T}}\mathbf{g}_e - \boldsymbol{\omega}_b \times \mathbf{v}_b$$

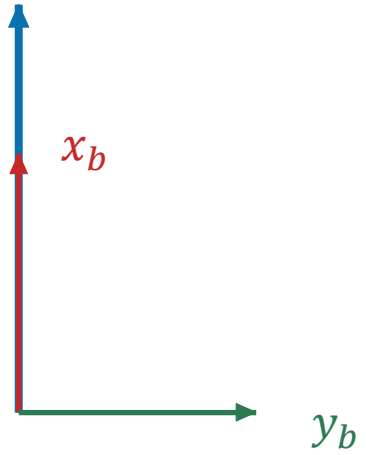
- **Physical meaning:** The final term comes from rotating coordinates, not extra external force. Even with constant Earth-frame velocity, forward and lateral body components can change.
- **Instantaneous check:** Body and NED axes are momentarily aligned. Velocity is 10 m/s north; the body yaws right at 1 rad/s. Then $\dot{\mathbf{v}}_b = [0, -10, 0]^{\mathsf{T}}\ \mathrm{m/s^2}$.

Body-frame component changes reflect both physical acceleration and rotation of the coordinate axes.

4.2.3 Tilt and Three-Dimensional Dynamics: Diagram and Check

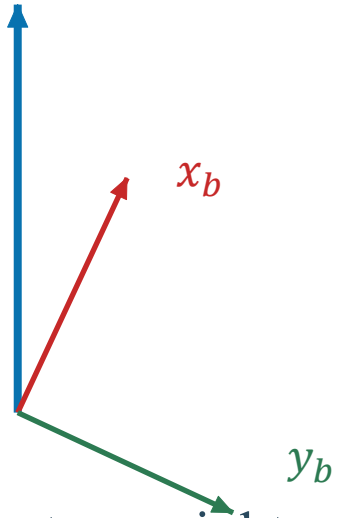
Northward Velocity as the Nose Turns Right

Northward velocity: 10 m/s



Initially aligned

Northward velocity: 10 m/s



Later: Nose turns right

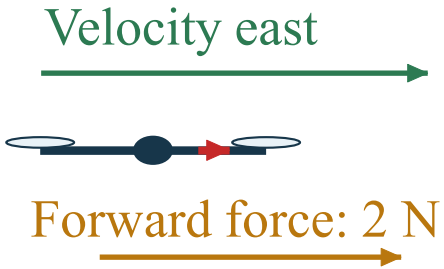
Initial instant: $\dot{\mathbf{v}}_e = 0, \dot{\mathbf{v}}_b = [0, -10, 0]^T$ m/s²

Figure: Fixed Earth-frame velocity and rotating body axes

Even with constant Earth-frame velocity, body-frame components change as the axes rotate.

4.2.3 Tilt and Three-Dimensional Dynamics: Combined Example: Earth Frame

Facing East: Earth-Frame Derivatives



- **Instantaneous conditions:** $m = 2\,\text{kg}$, $g = 9.8\,\text{m/s}^2$, $J = \text{diag}(0.02,0.03,0.04)\ \text{kg·m}^2$. Level and facing east, with $\mathbf{v}_b = [2,0,0]^{\mathrm{T}}$ m/s and $\boldsymbol{\omega}_b = [0,0,1]^{\mathrm{T}}$ rad/s.
- **Inputs:** $\mathbf{F}_{\mathrm{ng},b} = [2,0,-19.6]^{\mathrm{T}}$ N and $\mathbf{M}_b = [0,0,0.08]^{\mathrm{T}}$ N·m. Vertical force balances gravity; forward force becomes eastward force.

$$R = R_z(90^\circ) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

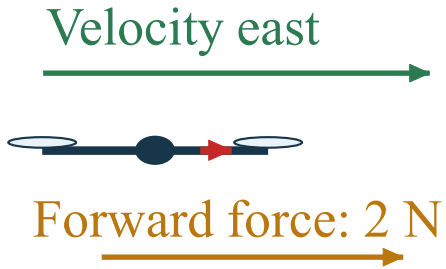
- **Attitude:** Yaw is 90°. Body forward points east and body right points south. R maps body components into NED.

$$\dot{\mathbf{r}}_e = R\mathbf{v}_b = [0,2,0]^{\mathrm{T}}\ \text{m/s}, \quad \dot{\mathbf{v}}_e = \mathbf{g}_e + R\mathbf{F}_{\mathrm{ng},b}/m = [0,1,0]^{\mathrm{T}}\ \text{m/s}^2$$

- **Result:** In Earth coordinates, the vehicle moves east at 2 m/s and accelerates east at 1 m/s². Net vertical force is zero.

4.2.3 Tilt and Three-Dimensional Dynamics: Combined Example: Body Frame

The Same Instant: Body and Attitude Derivatives



- **The same instantaneous example:** $J = \mathrm{diag}(0.02,0.03,0.04)\ \mathrm{kg}\cdot\mathrm{m}^2$, $\mathbf{v}_b = [2,0,0]^{\mathrm{T}}\ \mathrm{m/s}$ and $\boldsymbol{\omega}_b = [0,0,1]^{\mathrm{T}}\ \mathrm{rad/s}$; $\mathbf{F}_{\mathrm{ng},b} = [2,0,-19.6]^{\mathrm{T}}\ \mathrm{N}$ and $\mathbf{M}_b = [0,0,0.08]^{\mathrm{T}}\ \mathrm{N}\cdot\mathrm{m}$.

$$\boldsymbol{\omega}_b \times \mathbf{v}_b = [0,2,0]^{\mathrm{T}}\ \mathrm{m/s^2}, \quad \boldsymbol{\omega}_b \times (J\boldsymbol{\omega}_b) = \mathbf{0}$$

$$\dot{\mathbf{v}}_b = \mathbf{F}_{\mathrm{ng},b}/m + R^{\mathrm{T}}\mathbf{g}_e - \boldsymbol{\omega}_b \times \mathbf{v}_b$$

$$= [1,0,-9.8]^{\mathrm{T}} + [0,0,9.8]^{\mathrm{T}} - [0,2,0]^{\mathrm{T}} = [1,-2,0]^{\mathrm{T}}\ \mathrm{m/s^2}$$

$$\dot{\boldsymbol{\omega}}_b = J^{-1}[\mathbf{M}_b - \boldsymbol{\omega}_b \times (J\boldsymbol{\omega}_b)] = [0,0,2]^{\mathrm{T}}\ \mathrm{rad/s^2}$$

- **Attitude:** Let $c = \sqrt{2}/2$, with $\mathbf{q} = [c,0,0,c]^{\mathrm{T}}$ and derivative $\dot{\mathbf{q}} = [-c/2,0,0,c/2]^{\mathrm{T}}\ \mathrm{s^{-1}}$. All values refer to the same instant; the negative lateral derivative comes from rotating coordinates.

Components differ between frames, but transformed physical results must agree.

4.2.3 Tilt and Three-Dimensional Dynamics: Review the Coupling

Connect Attitude, Forces, and Motion

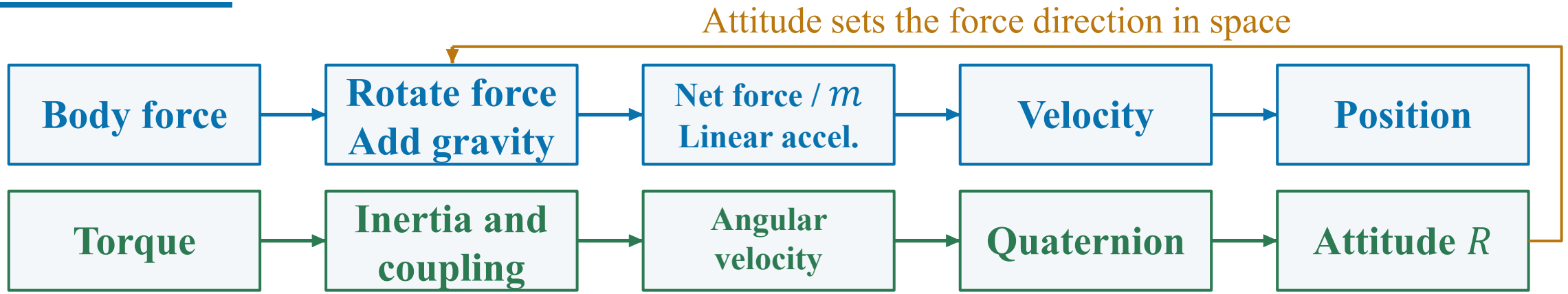


Figure: Two coupled chains of kinematics and dynamics

- **Conventions:** R maps forward-right-down (FRD) into NED. Count gravity once; retain coupling for body-velocity derivatives and nonspherical inertia.
- Initial states, parameters, inputs, and integration method set trajectories. Validate basic response analytically, numerical error by varying step size, and model accuracy by measurements.

First align coordinates and force conventions, then build the model with complete equations and interpretable examples.

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4.2.1 Hovering and Vertical Motion

Textbook: Sec. 4.1.4.1

4.2.2 Differential Thrust and Rotation

Textbook: Sec. 4.1.4.1

4.2.3 Tilt and Three-Dimensional Dynamics

Textbook: Secs. 4.1.3.2 and 4.1.4.1

4.2.4 Continuous Equations and Numerical Integration

Forward Euler, step-size error, and numerical stability

4.2.5 Analytical Checks and Exercises

Check balance, single-axis rotation, and error sources

4.2.4 Continuous Equations and Numerical Integration: Forward Euler

Turn Continuous Equations into Discrete-Time Calculations

- Store states every h. Derivatives describe continuous change between samples; integration approximates its accumulation.

$$\mathbf{x}(t+h) = \mathbf{x}(t) + \int_t^{t+h} f\left(\mathbf{x}(\tau), \tau\right) \mathrm{d}\tau$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + hf(\mathbf{x}_k, t_k), \qquad t_k = kh$$

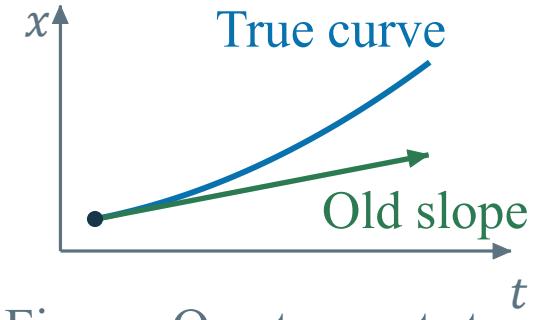


Figure: One tangent step

- **Synchronous updates:** Compute all derivatives from the same old state, then write all new states together. Using updated velocity for position changes the forward Euler algorithm and its error.
- **For sufficiently smooth equations:** Euler's local one-step error $\mathcal{O}(h^2)$; fixed-time accumulated error usually $\mathcal{O}(h)$. Smaller steps help, but cannot fix wrong coordinate directions, units, or force equations.

A discrete algorithm approximates continuous integration; update timing and step size jointly determine numerical error.

4.2.4 Continuous Equations and Numerical Integration: Two Hand-Calculated Steps

How Do Individual State Updates Build Motion?

- **Conditions:** Start vertical motion at $z_D = -1\,\text{m}$ and $v_D = 0$. Total thrust is 21.6 N, so $a_D = -1\,\text{m/s}^2$. Use forward Euler with $\Delta t = 0.1\,\text{s}$.

$$z_{D,k+1} = z_{D,k} + \Delta t\, v_{D,k}, \qquad v_{D,k+1} = v_{D,k} + \Delta t\, a_{D,k}$$

Both recurrences read the same old state, then write back together.

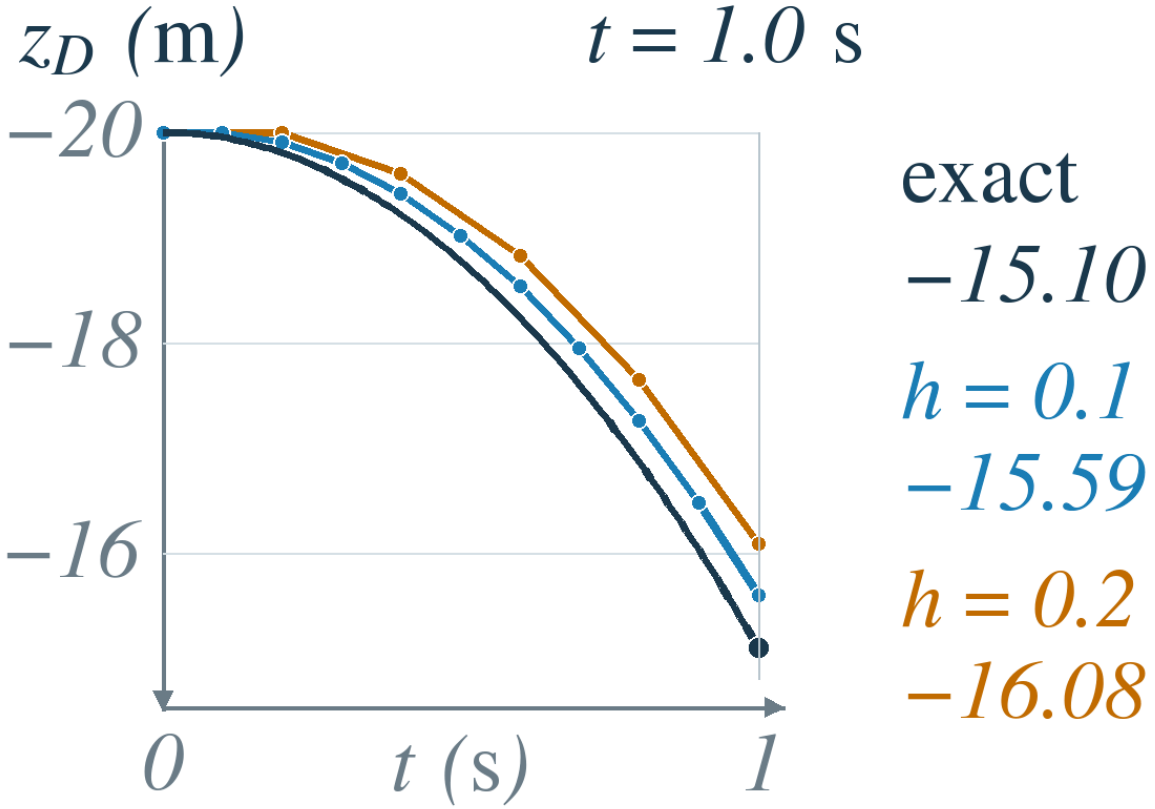
$t = 0$	$t = 0.1$	$t = 0.2$
$z_D = -1$	$z_D = -1$	$z_D = -1.01$
$v_D = 0$	$v_D = -0.1$	$v_D = -0.2$

First step: old velocity 0 leaves position unchanged; the continuous solution has risen 0.005 m.

- **Check at 1 s:** With $h = 0.1\,\text{s}$, the rise is 0.45 m; with $h = 0.01\,\text{s}$, it is 0.495 m. The analytical rise is 0.5 m. Both step sizes give the same final velocity; position error shrinks with step size.

4.2.4 Continuous Equations and Numerical Integration: Analytical Solution and Step-Size Error

Free Fall: Exact Velocity, Step-Dependent Position Error



Free fall from −20 m: analytical and two step sizes

Comparison at 1 s

Step size	Position, m	Fall deficit, m
Analytical	−15.10	0
0.1 s	−15.59	0.49
0.2 s	−16.08	0.98

All final velocities: 9.8 m/s

Constant acceleration: exact sampled velocity; position using old velocity underestimates fall:

$$z_E(t) - z(t) = -4.9\,t\,h$$

4.2.4 Continuous Equations and Numerical Integration: Attitude Integration Error

Normalization Restores Norm; Angle Error Remains

- **Setup:** $\mathbf{q}(0) = [1,0,0,0]^{\mathrm{T}}$; constant $\boldsymbol{\omega}_b = [0,0,\pi/2]^{\mathrm{T}}$ rad/s. Use forward Euler for 1 s with $h = 0.1$ s. Without normalization, norm reaches 1.031225 and violates the unit constraint.

$$\tilde{\mathbf{q}}_{k+1} = \mathbf{q}_k + h/2\,\mathbf{q}_k \otimes [0, \boldsymbol{\omega}_b^{\mathrm{T}}]^{\mathrm{T}}, \quad \mathbf{q}_{k+1} = \tilde{\mathbf{q}}_{k+1} / \| \tilde{\mathbf{q}}_{k+1} \|$$

- Divide by the norm each step to restore unit length. Final yaw 89.815627° is 0.184373° below the analytical 90°. This discrete-integration error still depends on step size.
- **Unit-quaternion finite steps give exact** attitude for constant body rates (Section 4.1, "Single-Axis Rotation and Finite Increments"). Time-varying rates still need approximation.

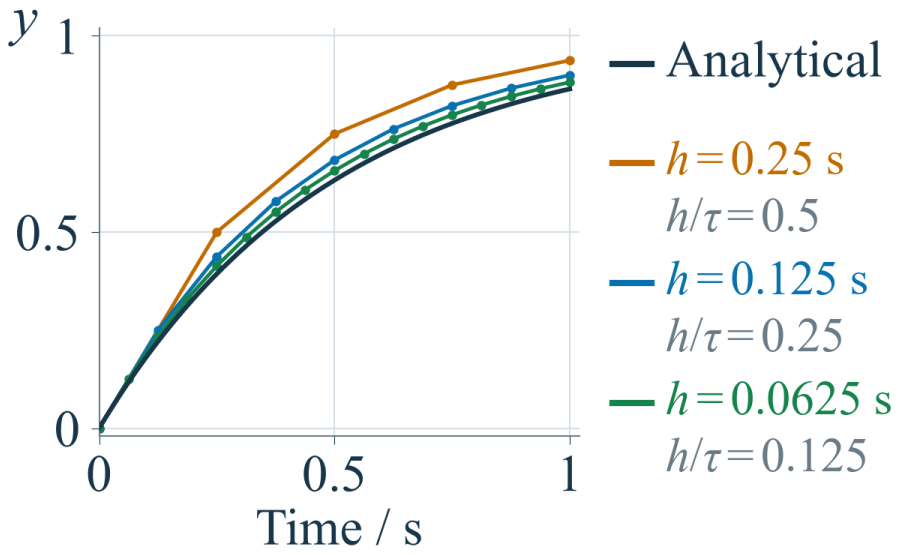


Figure: Enlarged yaw-angle detail

Unit norm ensures validity; attitude accuracy still depends on integration method and step size.

4.2.4 Continuous Equations and Numerical Integration: Dynamic-Response Accuracy

Match the Step Size to the Model's Time Scale



Step size vs. accuracy (teaching example)

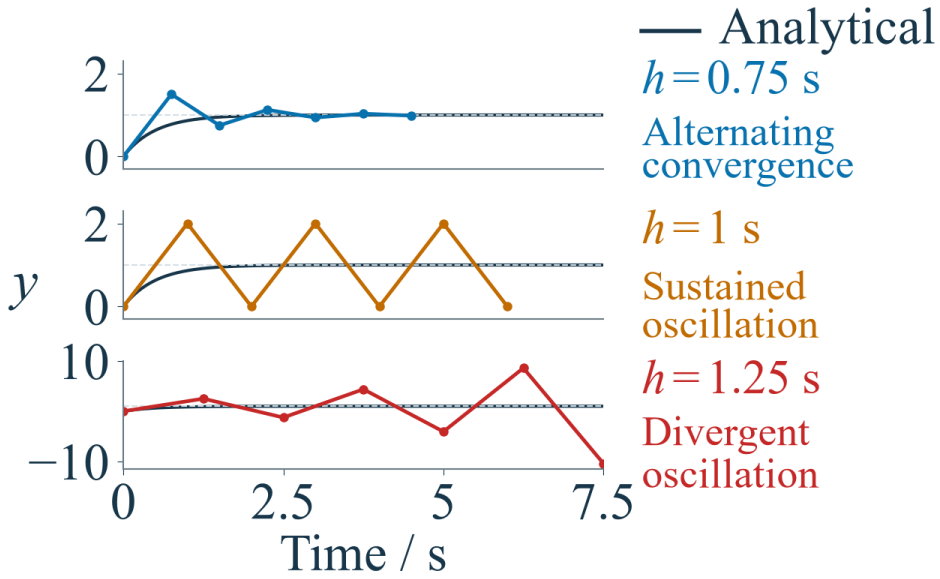
- **Conditions:** $\tau\dot{y} + y = u$, $\tau = 0.5$ s; start at 0, step input to 1. Time constant sets response speed.
- **Read at 1 s:** The analytical value is 0.8647. Euler values for 0.25, 0.125, and 0.0625 s steps are 0.9375, 0.8999, and 0.8819.
- **Choose a step:** Errors are 0.073, 0.035, and 0.017, approximately proportional to step size. Stability needs only $h < 2\tau$; accuracy requires choosing the step from the allowed error.

$$y_{k+1} = y_k + (h/\tau)(1 - y_k), \qquad y(t) = 1 - e^{-t/\tau}$$

The model's time scale sets how fast it changes; the allowed error sets how small the step must be.

4.2.4 Continuous Equations and Numerical Integration: Euler Stability Limits

A Stable Physical System Can Oscillate or Diverge Numerically



Three Euler responses (teaching example)

- **Error:** Initial value 0, unit step; define $e_k = y_k - 1$. Error multiplier: $1 - h/\tau$. Decay requires magnitude <1, hence $0 < h < 2\tau$.
- **Plots:** $\tau = 0.5$ s; step-size multipliers −0.5, −1, −1.5 give alternating convergence, sustained oscillation, divergence. Vertical scales differ.
- Stable, monotonic continuous response; recurrence causes oscillation. Stability is the minimum; also meet allowed error.

$$e_{k+1} = (1 - h/\tau)e_k, \qquad |1 - h/\tau| < 1$$

Integration steps can introduce numerical oscillation; numerical stability does not guarantee sufficient accuracy.

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4.2.4 Continuous Equations and Numerical Integration

Forward Euler, step-size error, and numerical stability

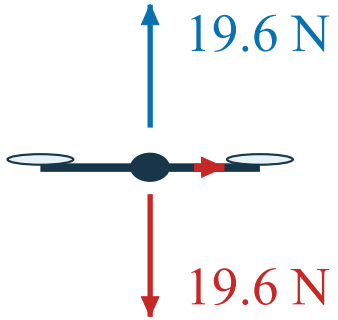
4.2.5 Analytical Checks and Exercises

Check balance, single-axis rotation, and error sources

4.2.5 Analytical Checks and Exercises: Zero Derivatives and Constant Torque

Two Baselines: Balance and Single-Axis Rotation

- **Balance baseline:** $m = 2\,\text{kg}$ and $g = 9.8\,\text{m/s}^2$. Initially level and stationary, with zero angular velocity. Upward thrust of 19.6 N and zero net torque make all state derivatives zero.
- **Perturbation:** Add 0.4 N thrust to obtain $a_D = -0.2\,\text{m/s}^2$. Zero derivatives at equilibrium do not imply a return after disturbance.



- **Single-axis baseline:** Separately set $J_x = 0.02\,\text{kg}\cdot\text{m}^2$ and $M_x = 0.04\,\text{N}\cdot\text{m}$, with zero initial angle and rate. Rotation about just one principal axis removes the coupling term.

$$\dot{p} = 2\,\text{rad/s}^2, \quad p(0.5) = 1\,\text{rad/s}, \quad \phi(0.5) = 0.25\,\text{rad}$$

- **Exercises:** Predict motion with 18 N thrust. Why is the single-axis response unchanged when inertia and torque both scale by 1.5?

Start with hand-solvable balance and single-axis cases to check signs, units, and basic responses.

4.2.5 Analytical Checks and Exercises: Physical Sources of Error

Equation Error or Integration Error?

- **Direction and units:** Upward free-fall acceleration indicates a wrong NED sign or gravity direction; 19.6 m/s² often means double-counted gravity. Degrees/s treated as radians/s gives an error factor near 57.3. Smaller steps cannot fix this.
- Halving the step and roughly halving endpoint error agree with first-order Euler convergence. If error does not fall, check equations, input timing, reference solution, and floating point.
- **Model error:** Accurate integration cannot fix omitted drag, wrong mass, or omitted actuator dynamics. Align inputs, parameters, and time before judging measured-trajectory differences.

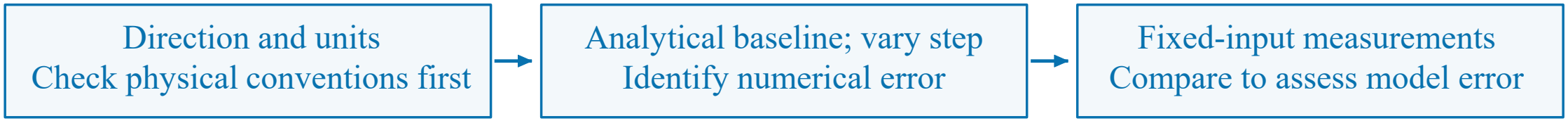


Figure: Physics → numerics → measurements (all examples are offline numerical calculations)

Use physical direction and analytical answers to locate problems before assessing integration accuracy or model fidelity.

4.2.5 Analytical Checks and Exercises: Reference Answers

Thrust Deficit and Proportional Inertia/Torque Scaling

- **Question 1:** A level vehicle at rest has mass $2\,\text{kg}$ and $g = 9.8\,\text{m/s}^2$, with only upward thrust $18\,\text{N}$ and weight. Thrust is $1.6\,\text{N}$ below weight, so acceleration is downward:

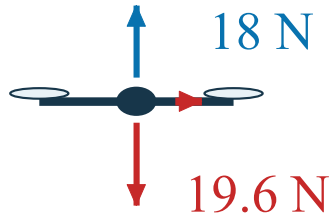
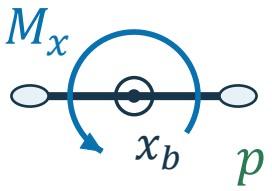


Figure: Translation / single-axis rotation

$$a_D = 9.8 - 18/2 = 0.8\,\text{m/s}^2$$

- **Question 2 (both ×1.5):** $J_x = 0.03\,\text{kg}\cdot\text{m}^2$ and $M_x = 0.06\,\text{N}\cdot\text{m}$, rotating only about principal axis x from rest. At $0.5\,\text{s}$, the result matches the original case:



$$\dot{p} = 0.06/0.03 = 2\,\text{rad/s}^2, \quad p(0.5) = 1\,\text{rad/s}, \quad \phi(0.5) = 0.25\,\text{rad}$$

- **Key:** $\dot{p} = M_x/J_x$ sets angular acceleration. Scale inertia and torque by 1.5: ratio stays 2, giving the same response from the same initial state, only about one principal axis.

Net load sets response direction; mass or inertia then determines its magnitude.

Classroom Exercises

Tilted Forces, One Quaternion Step, and Step Size

- **Exercise 1 (tilted forces):** $m = 2$ kg and $T = 20$ N, with nose-down pitch $\theta = -0.2$ rad and zero roll and yaw. Find northward and downward acceleration, and the direction of net vertical force.
- **Exercise 2 (one quaternion step):** $\mathbf{q}(0) = [1,0,0,0]^{\mathrm{T}}$ and $\boldsymbol{\omega}_b = [0,0,1]^{\mathrm{T}}$ rad/s. Take one forward Euler step with $h = 0.2$ s. Give $\mathbf{q}_1$ and normalized yaw; compare with the analytical 0.2 rad.
- **Exercise 3 (step selection):** A first-order element has $\tau = 0.2$ s. For h = 0.05, 0.3, and 0.5 s, calculate the error multiplier per step. Identify monotonic convergence, alternating convergence, or divergence.
- **Hints:** Use radians throughout. Quaternions are scalar first; calculate the increment as $(h/2)\,\mathbf{q} \otimes [0, \boldsymbol{\omega}_b^{\mathrm{T}}]^{\mathrm{T}}$. The first-order error multiplier is $1 - h/\tau$.

Reference Answers

Reference Answers: Forces, Quaternions, and Step Size

- **Exercise 1:** $a_N = -10\sin(-0.2) = +1.987\ \mathrm{m/s^2}$ (northward); $a_D = 9.8 - 10\cos 0.2 \approx -0.0007\ \mathrm{m/s^2}$. Vertical thrust slightly exceeds weight, giving a tiny upward net force. Height change still depends on vertical velocity.
- **Exercise 2:** $\mathbf{q}_1 = [1,0,0,0.1]^{\mathrm{T}}$, norm 1.00499. Normalization gives $[0.99504,0,0,0.09950]^{\mathrm{T}}$. Yaw $2\arctan 0.1 = 0.19934$ rad is 0.00066 rad (about 0.038°) below 0.2 rad.
- **Exercise 3:** Multipliers are 0.75, −0.5, and −1.5: 0.05 s converges monotonically, 0.3 s alternates and converges, and 0.5 s diverges. Stability requires $h < 2\tau = 0.4$ s; accuracy may require smaller h to meet the allowed error.

Section Summary

From Loads to State Updates and Analytical Checks

- **Method:** Divide net force by mass for linear acceleration; use Euler's dynamics for angular acceleration. R maps body force into NED. Evaluate all four state-equation groups at the same instant, then advance with forward Euler.
- **Key judgments:** Balance does not imply zero velocity. Body angular velocity may change with zero external torque. Quaternion normalization does not guarantee angle accuracy. Numerical stability does not guarantee sufficient accuracy.
- **Common errors:** NED D points down; upward is negative. Count gravity once in convention A or B. Keep degrees and radians distinct. Read all old states before writing all new states within a step.
- **Next:** Section 4.3 replaces known rotor thrusts with propulsion and force models. Section 4.5 implements the four equation groups and checks them against these analytical values.

Align coordinates and gravity conventions first; use analytical cases and step-size comparisons to check the model and integration separately.